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Non-perturbative equivalences among large N_c gauge theories with adjoint and bifundamental matter fields

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Abstract: We prove an equivalence, in the large N_c limit, between certain $\mathrm{U}(\boldsymbol{N}_c)$ gauge theories containing adjoint representation matter fields and their orbifold projections. Lattice regularization is used to provide a non-perturbative definition of these theories; our proof applies in the strong coupling, large mass phase of the theories. Equivalence is demonstrated by constructing and comparing the loop equations for a parent theory and its orbifold projections. Loop equations for both expectation values of single-trace observables, and for connected correlators of such observables, are considered; hence the demonstrated non-perturbative equivalence applies to the large $\boldsymbol{N}_c$ limits of both string tensions and particle spectra.

Keywords: 1/N Expansion, Lattice Gauge Field Theories.

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1. Introduction

Various examples are known in which two gauge theories with N_c colors, which differ for all finite N_c, become indistinguishable in the $N_c \to \infty$ limit. Such equivalences include lattice Yang-Mills theories with mixed fundamental-adjoint representation actions, whose leading large N_c limits coincide with those of pure fundamental representation actions (provided one suitably modifies the value of the lattice gauge coupling) [1, 2]. Another example is the volume independence of large N_c gauge theories. This is often referred to as Eguchi-Kawai reduction [3]; see refs. [4, 5, 6] for more recent discussions. Such large N_c equivalences can also involve theories with differing gauge groups. As a trivial example, the existence of the $N_c \to \infty$ limit in $\mathrm{U}(N_c)$ pure gauge theories implies that Yang-Mills theories with gauge groups $\mathrm{U}(N_c)$ and $\mathrm{U}(kN_c)$ (for any positive integer k) and coinciding 't Hooft couplings are indistinguishable as $N_c \to \infty$.

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Recently, possible large N_c equivalences between pairs of theories related by so-called "orbifold" projections have received attention [7]–[17]. In this context, orbifold projection is a technique for constructing a "daughter" theory, starting from some "parent" theory, by retaining only those fields which are invariant under a discrete symmetry group of the parent theory.[1] The basis for a possible large N_c equivalence between parent and daughter orbifold theories comes from the fact that in the large-N_c limit of ordinary perturbation theory, planar graphs of the orbifold theory exactly coincide with planar graphs of the original theory, up to a simple rescaling of the gauge coupling constant [11]. Because perturbation theory is only an asymptotic expansion, coinciding perturbative expansions do not imply that two theories must be equivalent. In particular, in asymptotically free theories the mass spectrum is purely non-perturbative, so coinciding perturbative expansions do not, by themselves, imply that parent and daughter orbifold theories have identical particle spectra. However, the existence of a perturbative equivalence between parent and daughter theories does make it natural to ask whether the large N_c equivalence is valid non-perturbatively. If true, there are a variety of interesting consequences [12, 13]. For example, the fact that supersymmetric theories may have non-supersymmetric orbifolds would imply that certain non-supersymmetric theories must develop an accidental boson-fermion degeneracy in part of their mass spectrum as $N_c \to \infty$.

To date, no non-perturbative proof of large N_c equivalence between parent and daughter orbifold theories has been given.[2] Several tests have been proposed to check whether the equivalence might hold non-perturbatively, both for supersymmetric [14], and non-supersymmetric [15, 16, 17] daughter theories. Evidence consistent with a possible non-perturbative large N_c equivalence has come from comparison of the holomorphic couplings in parent and daughter theories [14], as well as from a matrix model analysis of low-energy effective actions [16]. Various results [8, 9] on conformal field theories obtained from AdS/CFT duality (which is widely believed, but unproven) are also consistent with a non-perturbative large N_c equivalence.

However, evidence of large N_c *inequivalence* between certain parent and daughter orbifold theories has also been found. In particular, a mismatch between the number of instanton zero modes in small volume was found in ref. [15], and in ref. [17] it was argued that a compactified orbifold theory, unlike its parent, undergoes a phase transition at a non-zero value of the compactification radius. These previously considered examples all involve cases where the parent theory is supersymmetric, and no issues involving non-perturbative regularization have been addressed. Beyond these specific examples, it can only be said that it is not yet clear under what circumstances a non-perturbative equivalence does, or does not, hold.

In this paper, we will present a proof of the non-perturbative equivalence between the large N_c limits of a simple class of parent gauge theories and their orbifold projections.

[1]The name "orbifold" comes from string theory, where daughter theories of this type originate as low-energy world-volume descriptions of D-branes on space-time orbifolds [7]–[10].

[2]Excluding the case of pure Yang-Mills theories, where large N_c equivalence under orbifold projection is nothing more than a repackaged form of the above-mentioned equivalence between $\mathrm{U}(N_c)$ and $\mathrm{U}(kN_c)$ Yang-Mills theories in the $N_c \to \infty$ limit.

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Specifically, we will consider $\mathrm{U}(N_c)$ gauge theories containing either scalar or fermion matter fields (or both) transforming in the adjoint representation of the gauge group. In order to have a rigorous basis for making non-perturbative arguments, we will use a lattice formulation of our theories. Physical observables of interest, including the mass spectrum of the theory, can be extracted from correlation functions of Wilson loops, possibly decorated with insertions of adjoint representation matter fields. These correlation functions obey a closed set of loop equations in the large-N_c limit.[3] Our strategy will entail: (*i*) showing that the large N_c loop equations of parent and daughter orbifold theories, for the relevant class of observables, coincide after trivial rescaling of coupling constants, and (*ii*) arguing that this system of loop equations can, at least in the phase of the theory continuously connected to strong coupling and large mass, uniquely determine the resulting correlation functions.

In other words, we will argue that comparison of large N_c loop equations can, under appropriate conditions, provide a sufficient means for determining when two theories have coinciding large N_c limits. This idea is not new; essentially the same strategy has previously been used in discussions of fundamental-adjoint universality [1], and the Eguchi-Kawai reduction [3]. Our argument for the unique reconstruction of correlation functions based on their loop equations will be completely rigorous in the phase of the theory which is continuously connected to strong coupling and large mass (for the matter fields). The extent to which one can uniquely reconstruct correlation functions from their loop equations more generally is discussed further in section 2.

The paper is organized as follows. In section 2 we establish our notation and give a self-contained review of loop equations in pure $\mathrm{U}(N_c)$ gauge theory. This includes both loop equations for expectation values of single Wilson loops, as well as the extension to multi-loop connected correlators. In this section we also discuss the reconstruction of correlation functions from their loop equations. The extension to gauge theories with matter fields in the adjoint representation is discussed in section 3. A key ingredient is the introduction of an "extended" higher-dimensional lattice (with one new dimension for each matter field flavor) in such a way that Wilson loops decorated with arbitrary insertions of adjoint matter fields become isomorphic with simple loops on the extended lattice. This will allow us to formulate loop equations for theories containing adjoint matter fields in a compact and elegant form which closely mimics the loop equations of pure gauge theories. In section 4 we discuss orbifold projections of $\mathrm{U}(N_c)$ gauge theories with adjoint matter fields. In order to be self-contained, and establish notation, we first review what is meant by an orbifold projection, and then derive the loop equations in orbifold projected theories. We observe that the loop equations are exactly the same in the original theory and its orbifold projections provided that (*i*) observables in the two theories are appropriately identified, (*ii*) coupling constants of the two theories are suitably scaled, (*iii*) global symmetries used to define the orbifold projection are not spontaneously broken in the parent theory, and (*iv*) global symmetries of the daughter theory which cyclically permute equivalent gauge group factors are also not spontaneously broken. Section 5 discusses implications of

[3]In the case of pure gauge theories, these loop equations are sometimes called the Migdal-Makeenko equations [18].

this correspondence of loop equations between parent and daughter theories. At least in the strong-coupling/large-mass phase of both theories, we argue that this correspondence constitutes a non-perturbative proof of the large N_c equivalence of parent and daughter theories. Possible generalizations and extensions are also briefly discussed. For simplicity of presentation, only correlation functions of bosonic observables are considered in sections 3 and 4, but the extension to fermionic observables is sketched in appendix A. Appendix B describes (in a more pedagogical manner than in the main text) how one may see the presence of confinement and a mass gap in the iterative strong-coupling solution of the loop equations.

2. Pure gauge theory

2.1 Definitions

Let Λ denote a d-dimensional (euclidean spacetime) lattice, which may be either infinite or finite in extent. To define a $\mathrm{U}(N_c)$ lattice gauge theory, one associates a unitary matrix $u[\ell] \in \mathrm{U}(N_c)$ with every directed link $\ell \in \Lambda$ of the lattice. Links (and plaquettes, *etc.*) are oriented; we will use $\bar{\ell}$ to denote the opposite orientation of link ℓ, and $u[\bar{\ell}] \equiv u[\ell]^\dagger$. Lattice Yang-Mills theory may be defined by the probability measure

$$d\mu \equiv \frac{e^S}{Z}\, d\mu_0 \,, \tag{2.1}$$

where

$$d\mu_0 \equiv {\prod_{\ell\in\Lambda}}' \, du[\ell] \tag{2.2}$$

denotes the product of Haar measure for every (positively oriented) link of the lattice,

$$Z \equiv \int d\mu_0 \; e^S \tag{2.3}$$

is the usual the partition function, and

$$S \equiv \beta {\sum_{p\in\Lambda}}' \; \mathrm{Re}\,\mathrm{tr}\; u[\partial p] \tag{2.4}$$

is the standard Wilson action involving a sum over all (positively oriented) plaquettes in the lattice. Here, ∂p denotes the boundary of plaquette p and $u[\partial p]$ is the ordered product of link variables around this plaquette boundary. The prime on the product over links in the measure (2.2), and on the sum over plaquettes in the action (2.4), are indicators that only positively oriented links or plaquettes, respectively, are to be included. Subsequent unprimed sums over (various sets of) links or plaquettes should be understood as not having this restriction. The form (2.4), in which every plaquette contributes equally, is appropriate for regular, isotropic lattices. More generally, the contribution of a given plaquette may depend on its orientation (or location), in which case

$$S \equiv {\sum_{p\in\Lambda}}' \; \beta_p \; \mathrm{Re}\,\mathrm{tr}\, u[\partial p] \,, \tag{2.5}$$

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with β_p some specified weight associated with every plaquette. We will never introduce a lattice spacing explicitly; all dimensionful couplings should be understood as measured in lattice units.

Wilson loops are the basic observables of the theory. For any (directed) closed loop C contained in the lattice, let $u[C]$ denote the ordered product of link variables around the loop C (starting from an arbitrarily chosen site on the loop), and define

$$W[C] \equiv \frac{1}{N_c} \operatorname{tr} u[C] \,. \tag{2.6}$$

(The factor of $1/N_c$ is included so that expectation values of Wilson loops have finite, non-trivial large-N_c limits.)

2.2 Loop equations

Loop equations are, in effect, Schwinger-Dyson equations for the expectation values of Wilson loops (or their products) [18]–[22]. To generate these loop equations, it is convenient to define operators δ_ℓ^A which vary individual link variables. Specifically,

$$\delta_\ell^A \, u[\ell'] \equiv \delta_{\ell\ell'} \, t^A \, u[\ell] \,, \tag{2.7}$$

where $\{t^A\}$ are $N_c \times N_c$ basis matrices for the Lie algebra of $\mathrm{U}(N_c)$, normalized to satisfy $\operatorname{tr}\, t^A t^B = \frac{1}{2}\, \delta^{AB}$ and

$$\sum_{A=1}^{N_c^2} (t^A)_{ij} \, (t^A)_{kl} = \frac{1}{2} \, \delta_{il} \, \delta_{jk} \,. \tag{2.8}$$

Because link variables are unitary,

$$\delta_\ell^A \, u[\bar\ell'] = -\delta_{\ell\ell'} \, u[\bar\ell] \, t^A \,. \tag{2.9}$$

Invariance of the Haar measure implies that the integral of any variation vanishes,

$$\int d\mu_0 \; \delta_\ell^A \; (\text{anything}) = 0 \,. \tag{2.10}$$

Choosing 'anything' to be $e^S \, \delta_\ell^A \, (W[C])$, and summing over the Lie algebra index A (which will not be indicated explicitly), gives the identity

$$\Big\langle \delta_\ell^A \delta_\ell^A \, W[C] \Big\rangle + \Big\langle \Big(\delta_\ell^A \, W[C] \Big) {\sum_{p\in\Lambda}}' \, \frac{1}{2} \beta_p \left(\delta_\ell^A \;\operatorname{tr}\left(u[\partial p] + u[\partial p]^\dagger \right) \right) \Big\rangle = 0 \,. \tag{2.11}$$

This identity is easiest to visualize in cases where the loop C traverses the link ℓ only once. For such cases, the identity (2.8) implies that the first term is just $\frac{1}{2} N_c$ times the expectation value of $W[C]$. The second term of the identity (2.11) generates terms in which plaquettes which also traverse the link ℓ are 'stitched' into the loop C in all possible ways. If both the loop C and some plaquette boundary ∂p contain the (directed) link ℓ, then

$$\Big\langle \Big(\delta_\ell^A \, W[C] \Big) \Big(\delta_\ell^A \;\operatorname{tr}\left(u[\partial p] + u[\partial p]^\dagger \right) \Big) \Big\rangle = \frac{1}{2} \left[\langle W[(\partial p) C] \rangle - \langle W[(\overline{\partial p}) C] \rangle \right] . \tag{2.12}$$

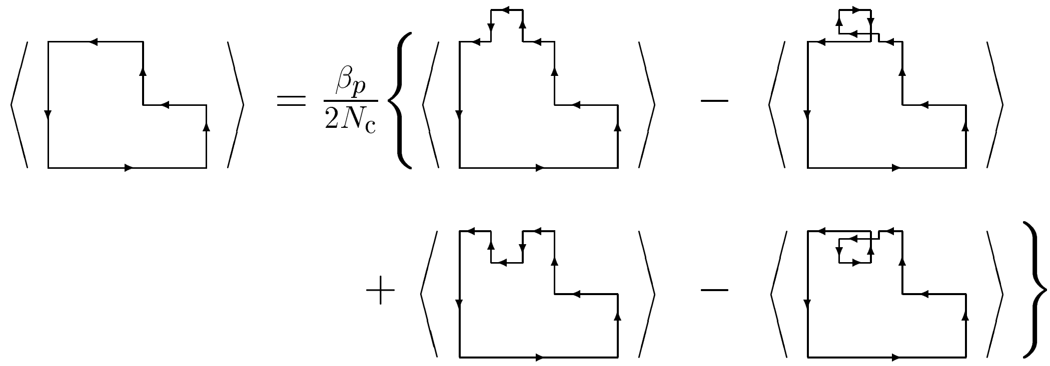

Figure 1: The loop equation for a non-self-intersecting loop in pure gauge theory, when only one link (in the middle of the top edge) is varied. In this example, the lattice is two-dimensional and the coupling β_p is the same for all plaquettes. To aid visualization, here and in the following figures, links which are multiply traversed are shown slightly offset. Arrows on the loop indicate the direction of traversal, not the orientation of lattice links. On the right-hand side of the equation, "untwisted" plaquettes are attached to the loop with a plus sign, and "twisted" plaquettes are attached with a minus sign.

For the concatenation of (∂p) and C to make sense, both loops C and (∂p) are to be regarded as starting with link ℓ [and hence $(\overline{\partial p})$ ends with link $\bar{\ell}$]. An example is shown in figure 1.

If the loop C traverses link ℓ more than once then there are additional contributions generated by the first term in (2.11) in which the loop is cut apart into two separate sub-loops. Loops which multiply traverse some link (in either direction) will be referred to as "self-intersecting".[4] As an example, if $C = \ell C' \bar{\ell} C''$, where C' and C'' are closed loops which do not contain link ℓ, then

$$\frac{1}{N_{\rm c}} \left\langle \delta_\ell^A \delta_\ell^A \, W[C] \right\rangle = \langle W[C] \rangle - \langle W[C'] \, W[C''] \rangle \,. \tag{2.13}$$

Similarly, if $C = C'C''$ where C' and C'' are non-self-intersecting closed loops both of which start with link ℓ, then

$$\frac{1}{N_{\rm c}} \left\langle \delta_\ell^A \delta_\ell^A \, W[C] \right\rangle = \langle W[C] \rangle + \langle W[C'] \, W[C''] \rangle \,. \tag{2.14}$$

These cases are illustrated in figure 2.

The identity (2.11) depends, by construction, on both the loop C and the choice of link which is varied. It will be simpler and more convenient to instead sum over the varied link ℓ. This will produce a single identity for each loop C, which we will call 'the loop equation for $W[C]$'. As discussed below, the resulting minimal set of loop equations will be sufficient to determine the expectation values of all Wilson loops in the strong coupling phase of the theory. Summing over all links in the original identity (2.11) yields the loop

[4] Hence, a loop may pass through a given site more than once and still be non-self-intersecting, provided it does not multiply traverse any link.

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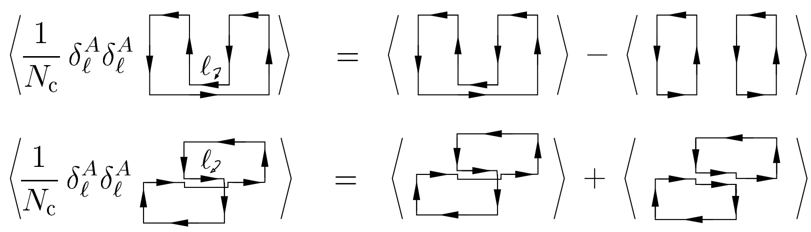

Figure 2: Second variations of $W[C]$, when the link ℓ being varied is traversed twice by the loop C in opposite directions (above) or in the same direction (below).

equation for the Wilson loop $W[C]$,

$$0 = \left\langle {\sum_{\ell\in\Lambda}}' \delta_\ell^A \delta_\ell^A \, W[C] \right\rangle + \left\langle {\sum_{\ell\in\Lambda}}' \Big(\delta_\ell^A \, W[C]\Big) {\sum_{p\in\Lambda}}' \, \frac{1}{2}\beta_p \left(\delta_\ell^A \; \mathrm{tr}\left(u[\partial p]{+}u[\partial p]^\dagger\right)\right) \right\rangle . \tag{2.15}$$

Non-zero contributions arise only for links ℓ which are traversed (in either direction) by the loop C and, in the second term, only for plaquettes whose boundaries also traverse the link ℓ. The first term (divided by N_c), now yields

$$\frac{1}{N_c} \left\langle {\sum_{\ell\in\Lambda}}' \, \delta_\ell^A \delta_\ell^A \, W[C] \right\rangle = \frac{1}{2}|C| \, \langle W[C]\rangle + \sum_{\mathrm{self-intersections}} \pm\langle W[C'] \, W[C'']\rangle \,, \tag{2.16}$$

where $|C|$ denotes the length (total number of links) of C, and C' and C'' are the two sub-loops produced by reconnecting C at a given self-intersection.[5] Here, and subsequently, the upper sign applies if the two traversals of the link at the intersection are in the same direction, and the lower sign if the traversals are in opposite directions. The second expectation in the identity (2.15) generates terms in which plaquettes which share any link with the loop C are 'stitched' into C in all possible ways. Combining these terms, the identity (2.15), divided by N_c, may be re-written as

$$\begin{aligned} \frac{1}{2}|C| \, \langle W[C]\rangle = & \sum_{\ell\subset C} \sum_{p|\ell\subset\partial p} \frac{\beta_p}{4N_c} \left[\langle W[(\overline{\partial p})C]\rangle - \langle W[(\partial p)C]\rangle\right] \\ & + \sum_{\mathrm{self-intersections}} \mp\langle W[C'] \, W[C'']\rangle \,. \end{aligned} \tag{2.17}$$

The sum over links $\ell \subset C$ includes all links contained in C, with each link oriented according to the direction in which it is traversed. If the loop multiply traverses some link, then the sum includes a separate term for each traversal. The plaquette sum includes plaquettes oriented such that their boundary contains link ℓ, not $\bar{\ell}$. The meaning of the first term on the right-hand side of (2.17) is again simple: for every link of the loop, "untwisted" plaquettes (in all possible directions) are attached with a plus sign, and "twisted" plaquettes with a minus sign, just like in figure 1.

[5]If loop C traverses a link ℓ (in either direction) K times, then the sum over self-intersections includes $K(K{-}1)/2$ terms generated by reconnecting each distinct pair of traversals of this link.

For any $\mathrm{U}(N_\mathrm{c})$ lattice gauge theory, the loop equation (2.17) is an exact identity relating the expectation value of any Wilson loop to the expectation values of modified loops with inserted plaquettes, and expectation values of products of loops generated by reconnecting the original loop at any self-intersections.

2.3 Large N_c limit, strong-coupling expansion, and uniqueness

If the plaquette weights β_p are scaled with the number of colors so that $\widetilde{\beta}_p \equiv \beta_p/N_\mathrm{c}$ is held fixed[6] as $N_\mathrm{c} \to \infty$, then the loop equation (2.17) becomes purely geometric with no explicit N_c dependence. It is known that the resulting large-N_c limit is a type of classical limit [23, 24] in which expectation values of products of Wilson loops factorize into single-loop expectations, up to corrections subleading in $1/N_\mathrm{c}$,

$$\langle W[C']\, W[C'']\rangle = \langle W[C']\rangle \, \langle W[C'']\rangle + O\left(\frac{1}{N_\mathrm{c}^2}\right). \tag{2.18}$$

Consequently, to leading order in the large N_c limit, the loop equations (2.17) become a set of closed, non-linear equations only involving expectation values of single loops [18],

$$\begin{aligned}\frac{1}{2}|C|\,\langle W[C]\rangle &= \sum_{\ell\subset C}\;\sum_{p|\ell\subset\partial p} \frac{1}{4}\,\widetilde{\beta}_p \left[\langle W[(\overline{\partial p})C]\rangle - \langle W[(\partial p)C]\rangle\right] \\ &\quad + \sum_{\mathrm{self-intersections}} \mp\langle W[C']\rangle\,\langle W[C'']\rangle + O\left(\frac{1}{N_\mathrm{c}^2}\right).\end{aligned} \tag{2.19}$$

This closed set of equations completely determines the expectation values of Wilson loops in the large-N_c limit, at least in the phase of the theory which is continuously connected to strong coupling (small $\widetilde{\beta}_p$). To prove this rigorously, it is sufficient to note that simply iterating the loop equations (2.19)[7] generates the lattice strong-coupling expansion — the expansion of expectation values in powers of the plaquette weights $\widetilde{\beta}_p$. The significance of this follows from the fact that the strong coupling expansion (unlike weak coupling perturbation theory) is known to have a non-zero radius of convergence [25, 26, 27]. Combined with the uniqueness of analytic continuation, this shows that the loop equations (2.19) uniquely determine the large-N_c expectation values of Wilson loops throughout the strong coupling phase of the theory.

In the large-N_c limit, all Wilson-action $\mathrm{U}(N_\mathrm{c})$ lattice gauge theories (even in finite volume) are believed to possess a third-order phase transition which is an artifact of the $N_\mathrm{c} \to \infty$ limit [28, 29, 5, 6]. This phase transition is driven by the behavior of the distribution of eigenvalues of elementary plaquettes $u[\partial p]$. For sufficiently strong coupling (small $\widetilde{\beta}$) the eigenvalue distribution is non-zero on the entire unit circle, while for sufficiently weak coupling (large $\widetilde{\beta}$) the support of this distribution lies only on a subset of the circle.

[6]This is the same as the usual 't Hooft scaling in which $g^2 N_\mathrm{c}$ (with g the continuum gauge coupling) is held fixed, since the lattice coupling $\beta_p \sim 1/g^2$.

[7]Starting with $\langle W[C]\rangle = 0$ for all loops, except the trivial zero-length loop which is unity, $\frac{1}{N_\mathrm{c}}\,\mathrm{tr}\,\mathbf{1} = 1$.

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On the weak-coupling side of this phase transition, it is sufficient (at least in simple models involving one, two, or three plaquettes [29]) to supplement the loop equations (2.19) with the trivial inequalities

$$|\langle W[C]\rangle| \leq 1\,, \tag{2.20}$$

in order to select the correct root of the loop equations.

2.4 Multi-loop connected correlators

The preceding formulation of loop equations may be easily extended to connected correlators involving a product of two or more Wilson loops. This extension will be needed for our later discussion, since the particle spectrum of a theory can only be extracted from two-loop correlators, not from single loop expectation values.

For notational convenience, we will define rescaled k-loop connected correlators,

$$\langle\langle W[C_1]W[C_2]\cdots W[C_k]\rangle\rangle \equiv N_{\rm c}^{2(k-1)}\,\langle W[C_1]W[C_2]\cdots W[C_k]\rangle_{\rm conn.} \tag{2.21}$$

The connected part of k-loop correlators vanish as $O(1/N_{\rm c}^{2(k-1)})$ relative to the totally disconnected part [24, 30]. Consequently, the overall factors of $N_{\rm c}$ inserted in (2.21) allow the rescaled connected correlators $\langle\langle W[C_1]\cdots W[C_k]\rangle\rangle$ to have finite, non-trivial large $N_{\rm c}$ limits.

For two-loop correlators, the natural generalization of the identity (2.15) is

$$\begin{aligned}
0 &= \int d\mu_0 \sum_{\ell\in\Lambda}{}' \,\delta^A_\ell \left\{ e^S\, \delta^A_\ell \left[\Big(W[C_1]-\langle W[C_1]\rangle\Big)\Big(W[C_2]-\langle W[C_2]\rangle\Big)\right]\right\} \\
&= \left\langle \sum_{\ell\in\Lambda}{}' \left[\delta^A_\ell\delta^A_\ell\, W[C_1] + \sum_{p\in\Lambda}{}' \,\frac{1}{2}N_{\rm c}\beta_p \Big(\delta^A_\ell\, W[C_1]\Big)\,\delta^A_\ell\Big(W[\partial p]+W[\overline{\partial p}]\Big)\right] \times \right. \\
&\qquad \left. \times \Big(W[C_2]-\langle W[C_2]\rangle\Big)\right\rangle + \left\langle \sum_{\ell\in\Lambda}{}' \Big(\delta^A_\ell\, W[C_1]\Big)\Big(\delta^A_\ell\, W[C_2]\Big)\right\rangle + (C_1 \leftrightarrow C_2)\,.
\end{aligned} \tag{2.22}$$

The expectation value in the second line of the result is only non-zero if loops C_1 and C_2 intersect (i.e., both loops traverse a common link); if so then this term is $1/N_{\rm c}$ times the expectation value of a single Wilson loop produced by reconnecting loops C_1 and C_2 at their mutual intersection(s). The disconnected part of the expectation in the first line of the result vanishes identically (both factors are zero), so what survives comes from connected two-loop correlators, as desired. The resulting contribution is also $O(1/N_{\rm c})$, due to the overall factor of $N_{\rm c}$ (either explicit, or hidden in the action of $\delta^A_\ell\delta^A_\ell$ on $W[C_1]$) multiplying an $O(1/N_{\rm c}^2)$ connected two-loop correlator.

Multiplying the identity (2.22) by an overall factor of $N_{\rm c}$ and sending $N_{\rm c}\to\infty$ yields

$$\begin{aligned}
\frac{1}{2}(|C_1|+|C_2|)\,\langle\langle W[C_1]\,W[C_2]\rangle\rangle &= \sum_{\ell\subset C_1}\ \sum_{p|\ell\subset\partial p} \frac{1}{4}\,\widetilde{\beta}_p \Big[\langle\langle W[(\overline{\partial p})C_1]\,W[C_2]\rangle\rangle - \\
&\qquad - \langle\langle W[(\partial p)C_1]\,W[C_2]\rangle\rangle\Big] +
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{\mathrm{self-intersections}\\(C_1)}} \mp \Big[\langle\langle W[C_1'] W[C_2] \rangle\rangle \, \langle W[C_1''] \rangle + \\
&\qquad\qquad\qquad\qquad + (C_1' \leftrightarrow C_1'') \Big] + \\
&+ \sum_{\mathrm{mutual-intersections}} \mp \frac{1}{2} \langle W[C_1 C_2] \rangle + O\left(\frac{1}{N_{\mathrm{c}}^2}\right) + \\
&+ (C_1 \leftrightarrow C_2) \,. \qquad (2.23)
\end{aligned}
$$

Once again, in each plaquette insertion term both the loop C_1 and the plaquette boundary ∂p are to be regarded as starting with link ℓ. In the mutual intersection terms, C_1 and C_2 are to be regarded as either starting with the intersection link or ending with its conjugate; the upper (lower) sign applies if both loops traverse the intersection link in the same (opposite) direction. The omitted $O(1/N_{\mathrm{c}}^2)$ piece involves the fully-connected three-loop correlator $\langle W[C_1'] W[C_1''] W[C_2] \rangle_{\mathrm{conn}}$. This may be dropped in the large-N_{c} limit because connected three-loop correlators vanish faster (by $1/N_{\mathrm{c}}^2$) than two-loop correlators.

The result (2.23) is a set of inhomogeneous linear equations for connected two-loop correlators. Just like the loop equations for single Wilson loops, these connected correlator loop equations may be solved iteratively (starting with all two-loop correlators equal to zero) to produce a strong coupling expansion with a non-zero radius of convergence. Hence, this set of equations (together with the single loop equations) completely determine two-loop connected correlators, at least in the phase of the theory which is continuously connected to strong coupling [and presumably beyond as well, when supplemented with the inequalities (2.20)].

The extension to higher multi-loop correlators is completely analogous, and will not be discussed explicitly.

3. Adjoint matter fields

3.1 Lattice discretization

At every site s of the lattice, we now add N_{s} independent scalar variables $\{\phi_a[s]\}$ ($a = 1, \cdots, N_{\mathrm{s}}$), and N_{f} pairs of fermionic variables $\{\psi_b[s], \bar{\psi}_b[s]\}$ ($b = 1, \cdots, N_{\mathrm{f}}$), all transforming in the adjoint representation of the $\mathrm{U}(N_{\mathrm{c}})$ gauge group. The scalars $\phi_a[s]$ are complex $N_{\mathrm{c}} \times N_{\mathrm{c}}$ matrices,[8] while the fermions $\psi_b[s]$ and $\bar{\psi}_b[s]$ are $N_{\mathrm{c}} \times N_{\mathrm{c}}$ matrices of independent Grassmann variables. Although we were more general in the last section, henceforth we will assume that the lattice Λ is a simple cubic lattice.

The measure for the theory has the usual form (2.1), where the decoupled measure $d\mu_0$ is now the product of Haar measure for every link variable and independent flat measures

[8] Since the adjoint representation is a real representation, one could introduce scalar variables as $N_{\mathrm{c}} \times N_{\mathrm{c}}$ hermitean matrices. We choose to use complex scalars instead, so that the resulting theory will have a $\mathrm{U}(N_{\mathrm{s}})$ global symmetry. This will allow us to apply non-trivial orbifold projections (even when $N_{\mathrm{s}} = 1$), as discussed in the next section.

for all the scalar and Grassmann variables,[9]

$$d\mu_0 = \left(\prod_{\ell\in\Lambda}{}' \, du[\ell] \right) \left(\prod_{s\in\Lambda} \prod_{a=1}^{N_{\rm s}} d\phi_a[s] \, d\phi_a^\dagger[s] \right) \left(\prod_{s\in\Lambda} \prod_{b=1}^{N_{\rm f}} d\psi_b[s] \, d\bar\psi_b[s] \right) . \tag{3.1}$$

The action is the sum of the pure-gauge Wilson action (2.5), which we now denote as $S_{\rm gauge}$, plus matter field contributions,

$$S = S_{\rm gauge} + S_{\rm scalar} + S_{\rm fermion} \, . \tag{3.2}$$

The scalar action will have the natural nearest-neighbor coupling plus some local potential energy,

$$S_{\rm scalar} = N_{\rm c}^2 \left\{ \frac{\kappa}{2} \sum_{\ell=\langle ss'\rangle\in\Lambda} \frac{{\rm tr}\left(\phi_a^\dagger[s]\, u[\ell]\, \phi_a[s']\, u[\bar\ell]\right)}{N_{\rm c}} - \sum_{s\in\Lambda} V\left[\frac{{\rm tr}\left(\phi_a^\dagger[s]\, \phi_a[s]\right)}{N_{\rm c}} \right] \right\} . \tag{3.3}$$

The notation $\ell = \langle ss' \rangle$ means that ℓ is the link which runs from site s to neighboring site s'; this sum runs over both orientations of every link. There is an implied sum over the repeated "flavor" index a. We have chosen the scalar action to have a $\mathrm{U}(N_{\rm s})$ global symmetry; the specific form of the scalar potential could be generalized at the cost of extra notational complexity. To ensure integrability of the full measure $d\mu$, the potential $V[\chi]$ should rise unboundedly for large arguments. For later convenience, we have inserted factors of $N_{\rm c}$ so that both the "hopping parameter" κ and the functional form of $V[\chi]$ may be kept fixed as $N_{\rm c} \to \infty$ (with $N_{\rm s}$ fixed).

The fermion action is

$$S_{\rm fermion} = N_{\rm c} \left\{ \frac{\kappa}{2i} \sum_{\ell=\langle ss'\rangle\in\Lambda} {\rm tr}\left(\bar\psi_b[s]\, \eta[\ell]\, u[\ell]\, \psi_b[s']\, u[\bar\ell]\right) - m \sum_{s\in\Lambda} {\rm tr}\left(\bar\psi_b[s]\, \psi_b[s]\right) \right\} , \tag{3.4}$$

with an implied sum on the flavor index b. We have chosen both scalars and fermions to have a common hopping parameter κ. This may always be arranged by suitably rescaling the scalar (or fermion) variables. Similarly, as long as the common fermion bare mass m is non-zero, it may be set to unity by an appropriate rescaling of variables. Physical quantities only depend on the ratio κ/m; hence large mass m is equivalent to small hopping parameter κ. In the fermion action (3.4), $\eta[\ell]$ is an imaginary phase factor assigned to each link in such a way that the product of these phases around every plaquette is minus one, $\eta[\partial p] = -1$. We will refer to it as the "fermion flavor connection"; as with any unitary connection, $\eta[\bar\ell] \equiv \eta[\ell]^\dagger$. A specific realization is

$$\eta[\ell] = i \prod_{\nu<\mu} (-1)^{x_\nu} \, , \tag{3.5}$$

if ℓ is the link running in the $\hat e_\mu$ direction starting from the site with coordinates x_μ.

[9] $d\phi_a[s] d\phi_a^\dagger[s]$ should be understood as denoting independent integration over each of the $2N_{\rm c}^2$ real degrees of freedom contained in $\phi[s]$. Similarly, $d\psi_b[s]$ and $d\bar\psi_b[s]$ should be understood as denoting independent integration over each of the $N_{\rm c}^2$ Grassmann degrees of freedom contained in $\psi_b[s]$ and $\bar\psi_b[s]$, respectively.

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The choice (3.4) for discretizing fermion fields is known as "staggered lattice fermions" [31, 32]; it has the virtue of being notationally compact and not cluttering expressions with extraneous Dirac indices and gamma matrices.[10] We have chosen the fermion action to have a $\mathrm{U}(N_{\mathrm{f}})$ global symmetry; this assumption could be relaxed and the bare mass m replaced by an arbitrary mass matrix at the cost of extra notational complexity. We have again inserted factors of N_{c} in a manner which will prove convenient when taking the large N_{c} limit (with N_{f} fixed).

3.2 Geometric encoding of observables

For theories with adjoint representation matter fields, the natural gauge invariant observables are Wilson loops "decorated" with arbitrary insertions of matter fields at sites through which the loop passes. To formulate appropriate generalizations of loop equations, a required first step is adopting some scheme for unambiguously labeling arbitrarily decorated loops.

Consider, for the moment, a theory with only fermionic matter fields. One possibility would be to define

$$W[\Gamma_1,\Gamma_2,\cdots,\Gamma_K]_{b_1\bar{b}_2\cdots b_K} = \frac{1}{N_{\mathrm{c}}}\,\mathrm{tr}\left(\psi_{b_1}[s_1]\,u[\Gamma_1]\,\bar{\psi}_{b_2}[s_2]\,u[\Gamma_2]\cdots u[\Gamma_{K-1}]\,\psi_{b_K}[s_K]\,u[\Gamma_K]\right), \tag{3.6}$$

etc. The Γ_i are (in general) arbitrary open paths on the lattice which, when concatenated, form a closed loop, with s_i the site at which segment Γ_i begins. Employing this notation is possible, but (a) its excessively lengthy, (b) it does not uniquely label observables (due to trace cyclicity), and (c) one can do better.

A more concise, geometric labeling of observables may be formulated if one considers a $(d{+}N_{\mathrm{f}})$ dimensional lattice constructed by tensoring the original lattice Λ with N_{f} copies of the integers, $\Lambda' \equiv \Lambda\times\mathbb{Z}^{N_{\mathrm{f}}}$. Each d-dimensional 'slice' of Λ' looks just like Λ, except that N_{f} additional (oriented) links, pointing into the N_{f} new dimensions, now emanate from every site. The basic idea is to treat the fermion variables $\{\bar{\psi}_b[s]\}$ as the connection associated with links pointing into the new dimensions, and $\{\psi_b[s]\}$ as the conjugate connection associated with the oppositely directed links. The connection on links pointing in directions lying in any of the original d dimensions is the initial gauge field $u[\ell]$. This is illustrated in figure 3. To write this more explicitly, let $\hat{e}_b$ denote unit vectors pointing in each of the N_{f} new dimensions. Links $\ell' \in \Lambda'$ either point in a direction which lies in the original d dimensions, in which case they may be labeled as $\ell' = (\ell,\vec{n})$ [with $\ell \in \Lambda$ and $\vec{n} \in \mathbb{Z}^{N_{\mathrm{f}}}$], or they point in one of the new directions in which case they may be labeled as $\ell' = (s,\vec{n},\pm\hat{e}_b)$ [with $s \in \Lambda$ and $\vec{n} \in \mathbb{Z}^{N_{\mathrm{f}}}$]. Let $\mathbb{Z}^{N_{\mathrm{f}}}_{+}$ denote the even sub-lattice of $\mathbb{Z}^{N_{\mathrm{f}}}$ (points whose coordinates sum to an even integer), and $\mathbb{Z}^{N_{\mathrm{f}}}_{-}$ the odd sub-lattice. We define a lattice link

[10] In d spacetime dimensions, the naive discretization of a single Dirac fermion is equivalent to $2^{\lfloor d/2\rfloor}$ species of staggered fermions [32] (with $\lfloor d/2\rfloor$ denoting the integer part of $d/2$).

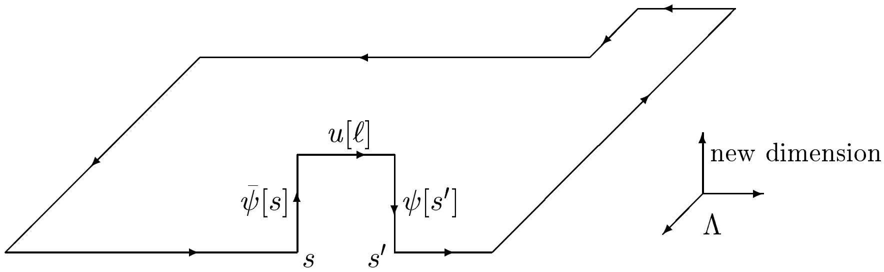

Figure 3: A closed loop in the extended lattice Λ', for the case of one fermion flavor. As indicated, links pointing in the new dimension represent fermion variables $\bar{\psi}[s]$ and $\psi[s']$. The observable associated with this closed loop is $\frac{1}{N_c}$ tr $\left(\bar{\psi}[s]\, u[\ell]\, \psi[s']\, u[\Gamma]\right)$, with ℓ the link running from site s to site s', and Γ denoting the portion of the contour lying in the original lattice Λ and running from site s' back to s. Only those decorated Wilson loops in which the number of ψ and $\bar{\psi}$ insertions coincide (for each flavor) form closed loops on Λ'.

variable $v[\ell']$ on Λ' such that

$$v[\ell'] = \begin{cases} u[\ell]\,, & \text{if } \ell' = (\ell, \vec{n}); \\ \bar{\psi}_b[s]\,, & \text{if } \ell' = (s, \vec{n}, +\hat{e}_b); \\ \psi_b[s]\,, & \text{if } \ell' = (s, \vec{n}, -\hat{e}_b). \end{cases} \tag{3.7}$$

Now apply the standard definition of Wilson loops, using the connection (3.7), to arbitrary closed paths in the lattice Λ'. More precisely, we define

$$W[C] = \pm \text{ tr } v[C]\,, \tag{3.8}$$

with the upper sign (+) applying if the path C is written as a sequence of links starting at a site in the even sub-lattice $\mathbb{Z}_+^{N_f}$, and the lower sign (−) if the path C is written as a sequence of links starting at a site in the odd sub-lattice $\mathbb{Z}_-^{N_f}$.[11] Each loop $C \in \Lambda'$ generates an observable resembling the example (3.6) (or else a normal Wilson loop if C lies entirely in a d-dimensional slice parallel to Λ), up to an overall sign. Note, however, that

$$v[\ell']\, v[\bar{\ell}'] \neq 1 \qquad \text{if } \ell' = (s, \vec{n}, \pm\hat{e}_b)\,, \tag{3.9}$$

because the connection $v[\ell']$ is not unitary for links pointing in the N_f new directions. Hence, backtracking "stubs" involving the new links do not cancel. This is illustrated in figure 4.

Of course, a path C in the extended lattice Λ' which includes M links pointing in the $+\hat{e}_b$ direction must also include M links pointing in the $-\hat{e}_b$ direction if it is to form a

[11] If the overall $\pm$ sign were omitted then, due to the Grassmann nature of fermion variables, different choices for the starting site of a loop would correspond to observables with differing overall signs. (For observables with an even number of fermion insertions, moving a Grassmann variable from one end of the trace to the other requires an odd number of interchanges with other Grassmann variables.) With the definition (3.8), the observable $W[C]$ depends only on the geometry of the loop $C \in \Lambda'$, not on the arbitrary choice of starting site.

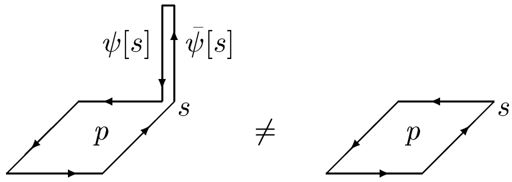

Figure 4: Backtracking "stubs" extending in the new directions do not cancel, since the associated "link variables" are non-unitary matter fields. The indicated loop on the left represents the observable $\frac{1}{N_c}$ tr $(\bar\psi[s]\,\psi[s]\,u[\partial p])$, where p is a plaquette whose boundary passes through the site s.

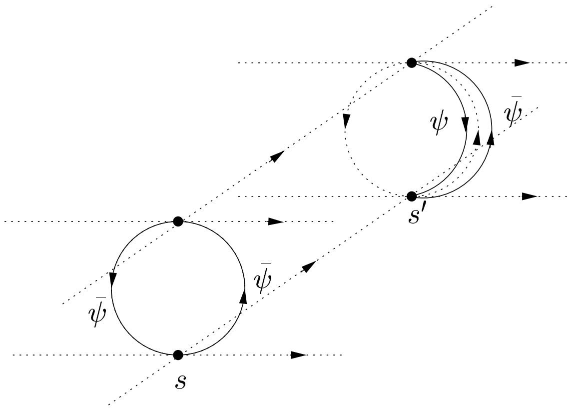

Figure 5: Examples of closed loops on the minimally extended lattice $\bar\Lambda_{\rm f}$, for the case of one fermion flavor. Dotted lines represent positively oriented links. The observables associated with the indicated closed paths are $\frac{1}{N_c}$ tr $(\bar\psi[s]\bar\psi[s])$ and $\frac{1}{N_c}$ tr $(\bar\psi[s']\psi[s'])$. Both are closed loops on $\bar\Lambda_{\rm f}$, but only the second one would be a closed loop on Λ'. All Wilson loops containing an even total number of fermion insertions form closed loops on $\bar\Lambda_{\rm f}$.

closed loop. So this mapping of loops in the higher-dimensional lattice Λ' onto observables of the form (3.6) only generates observables which separately conserve the number of each staggered fermion species. This is adequate for some purposes, but it is insufficient if one wishes to consider theories with non-diagonal or Majorana mass terms. More importantly, it is inadequate even in theories where net fermion number of each species is conserved, if one wishes to consider two (or higher) point correlation functions in all possible flavor symmetry channels.

One may accommodate a larger class of observables by appropriately identifying sites in Λ', since this enlarges the set of closed loops. The smallest lattice, and the largest set of acceptable observables, is produced by identifying all sites in Λ' which differ by even translations in $\mathbb{Z}^{N_{\rm f}}$ (those whose displacement vectors lie in $\mathbb{Z}_+^{N_{\rm f}}$). The result is a lattice $\bar\Lambda_{\rm f} \equiv \Lambda'/\mathbb{Z}_+^{N_{\rm f}}$ whose sites are just $\Lambda \otimes Z_2$, but where each site is connected to its Z_2 partner by $N_{\rm f}$ distinct, positively oriented links. This is illustrated in figure 5. Let $\widetilde\Lambda$ denote the Z_2 image of the original sublattice Λ, and let $\tilde s$ (or $\tilde\ell$ or $\tilde p$) denote the Z_2 partner of any site s (or link ℓ or plaquette p). The reduction of the definition (3.7) of the lattice link

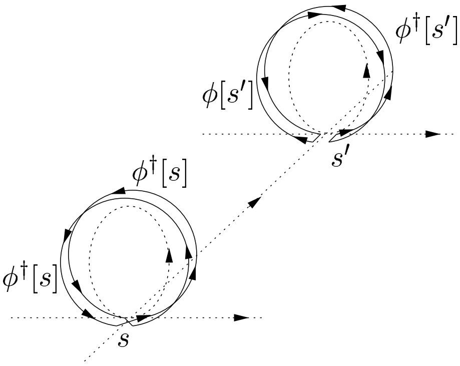

Figure 6: Examples of closed loops on the extended lattice $\bar{\Lambda}_{\rm s}$ for the case of one scalar flavor. Dotted lines represent (oriented) links. The observables associated with the indicated closed paths are $\frac{1}{N_{\rm c}}\,{\rm tr}\left(\phi^\dagger[s]\phi^\dagger[s]\right)$ and $\frac{1}{N_{\rm c}}\,{\rm tr}\left(\phi^\dagger[s']\phi[s']\right)$. Wilson loops with any number of scalar insertions form closed loops on $\bar{\Lambda}_{\rm s}$.

variable is[12]

$$v[\ell'] = \begin{cases} u[\ell]\,, & \text{either } \ell' = \ell \in \Lambda\,, \quad \text{or } \ell' = \tilde{\ell} \in \widetilde{\Lambda}\,; \\ \bar{\psi}_b[s]\,, & \text{either } \ell' = (s, +\hat{e}_b)\,, \quad s \in \Lambda\,, \quad \text{or } \ell' = (\tilde{s}, +\hat{e}_b)\,, \quad \tilde{s} \in \widetilde{\Lambda}\,; \\ \psi_b[s]\,, & \text{either } \ell' = (s, -\hat{e}_b)\,, \quad s \in \Lambda\,, \quad \text{or } \ell' = (\tilde{s}, -\hat{e}_b)\,, \quad \tilde{s} \in \widetilde{\Lambda}. \end{cases} \tag{3.10}$$

All gauge invariant bosonic observables of the form (3.6) (i.e., those in which the total number of fermion insertions is even) may now be represented by closed loops in $\bar{\Lambda}_{\rm f}$.[13] Gauge invariant fermionic observables do not correspond to closed loops in the extended lattice $\bar{\Lambda}_{\rm f}$, but rather to open paths whose endpoints are Z_2 images of each other — paths beginning at some site s and ending at $\tilde{s}$. As will be seen shortly, large $N_{\rm c}$ loop equations for either expectation values or multi-loop correlators of bosonic observables will not involve fermionic observables. For simplicity, we will focus on the treatment of bosonic observables in the following discussion, and relegate discussion of correlators of fermionic observables to appendix A.

"Gauge-fermion" plaquettes (i.e., plaquettes containing both fermion links and ordinary gauge links) reproduce the hopping terms in the fermion action (3.4). For the plaquette p whose boundary contains the gauge link $\bar{\ell} = \langle s's\rangle \in \Lambda$ and the fermion link $(s, +\hat{e}_b)$,

$${\rm tr}\left(v[\partial p]\right) = {\rm tr}\left(\bar{\psi}_b[s]\, u[\ell]\, \psi_b[s']\, u[\bar{\ell}]\right), \tag{3.11}$$

which coincides (up to the phase $\eta[\ell]$) with the hopping term in the action (3.4).

The above strategy may be applied equally well to theories with adjoint representation scalars. With scalar insertions, there are no subtleties concerning the overall sign of an

[12] $(s, +\hat{e}_b)$ denotes the positively oriented link which runs from site s to $\tilde{s}$ in the direction $+\hat{e}_b$, and $(\tilde{s}, -\hat{e}_b)$ denotes the same link but in its opposite orientation. $(\tilde{s}, +\hat{e}_b)$ denotes the positively oriented link which runs from site $\tilde{s}$ to s in the direction $+\hat{e}_b$, and $(s, -\hat{e}_b)$ denotes the same link in its opposite orientation.

[13] If $\widetilde{C}$ denotes the Z_2 image of a loop C (so that every link ℓ in C is replaced by its Z_2 partner $\tilde{\ell}$), then the corresponding observables differ only by an overall sign, $W[\widetilde{C}] = -W[C]$. Hence, the set of all closed loops on $\bar{\Lambda}_{\rm f}$ represents all single-trace bosonic observables as well as their negations.

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observable. Hence, one may divide the covering lattice Λ' by the entire $\mathbb{Z}^{N_s}$ translation group. The net result is a lattice $\bar{\Lambda}_s$ whose sites coincide with those of Λ but where every site now has N_s independent oriented links which connect the site to itself, as illustrated in figure 6. The appropriate lattice link variable is now[14]

$$v[\ell'] = \begin{cases} u[\ell]\,, & \ell' = \ell \in \Lambda; \\ \phi_a^\dagger[s]\,, & \ell' = (s, +\hat{e}_a), \quad s \in \Lambda; \\ \phi_a[s]\,, & \ell' = (s, -\hat{e}_a), \quad s \in \Lambda. \end{cases} \tag{3.12}$$

"Gauge-scalar" plaquettes (those containing both scalar and gauge links) now reproduce the hopping terms in the scalar action (3.3).

By combining the above extensions of the underlying lattice, one may, of course, consider a theory containing both fermions and scalars. The net result is that all bosonic single-trace gauge invariant observables in the original lattice gauge theory with adjoint representation scalars and/or fermions may be represented by Wilson loops defined on a minimally extended lattice $\bar{\Lambda}$.

3.3 Loop equations

Extending the previous derivation of loop equations to lattice gauge theories containing adjoint representation matter fields is straightforward. We will use the following shorthand for derivatives with respect to matter fields,

$$\delta_{s,a}^A \equiv (t^A)_{ij} \frac{\delta}{\delta\phi_a[s]_{ij}}\,, \qquad \bar{\delta}_{s,a}^A \equiv (t^A)_{ij} \frac{\delta}{\delta\phi_a^\dagger[s]_{ij}}\,, \tag{3.13}$$

$$\delta_s^{A,b} \equiv (t^A)_{ij} \frac{\delta}{\delta\psi_b[s]_{ij}}\,, \qquad \bar{\delta}_s^{A,b} \equiv (t^A)_{ij} \frac{\delta}{\delta\bar{\psi}_b[s]_{ij}}\,, \tag{3.14}$$

so that $\delta_{s,a}^A\, \phi_{a'}[s'] = \delta_{ss'}\, \delta_{aa'}\, t^A$, *etc.* Because our matter fields are not unitary, these variations effectively replace a matter field by a Lie algebra generator, rather than (left) multiplying by a generator [c.f. eq. (2.7)]. It remains true that the integral with the decoupled measure $d\mu_0$ of the variation of anything vanishes. Defining

$$\Delta \equiv \frac{1}{N_c}\, e^{-S} \left\{ {\sum_{\ell\in\Lambda}}' \delta_\ell^A\, e^S\, \delta_\ell^A - \sum_{s\in\Lambda} \left[\bar{\delta}_{s,a}^A\, e^S\, \delta_{s,a}^A + \delta_{s,a}^A\, e^S\, \bar{\delta}_{s,a}^A + \bar{\delta}_s^{A,b}\, e^S\, \delta_s^{A,b} - \delta_s^{A,b}\, e^S\, \bar{\delta}_s^{A,b} \right] \right\}, \tag{3.15}$$

a natural generalization of the previous loop equation is simply

$$0 = \langle \Delta\, \mathcal{O} \rangle \tag{3.16}$$

for any observable $\mathcal{O}$. The motivation for the choice of signs in the definition (3.15) is discussed below, after eq. (3.23).

As just discussed, single trace bosonic observables (i.e., decorated Wilson loops) may be associated with closed loops in the extended lattice $\bar{\Lambda}$. If the observable $\mathcal{O}$ under

[14] $(s, +\hat{e}_a)$ denotes the "scalar" link which connects site s to itself running in direction $\hat{e}_a$; $(s, -\hat{e}_a)$ is the opposite orientation of the same link. (If we had chosen our scalar variables to be hermitean, then it would have been natural to regard these links as unoriented.)

consideration is $W[C] \equiv \frac{1}{N_c}$ tr $v[C]$ for some loop $C \in \bar{\Lambda}$, then $\Delta\mathcal{O}$ will be a sum of (a) terms proportional to $W[C]$ itself, (b) decorated loops $W[C']$ where C' is a deformation of the loop C produced by inserting a plaquette (in the extended lattice $\bar{\Lambda}$), and (c) products of loops $W[C']\,W[C'']$ (on the extended lattice) produced by reconnecting $W[C]$ at self-intersections. We discuss each of these three pieces in turn.

Terms proportional to $W[C]$ are generated in $\Delta W[C]$ when both link variations δ^A_ℓ act on the same link matrix in $W[C]$ (as seen earlier). But such terms will now also be generated when a scalar variation $\bar{\delta}^A_{s,a}$ (or $\delta^A_{s,a}$) acts on the local potential part of the scalar action. This will bring down a factor of $\phi_a[s]$ (or $\phi^\dagger_a[s]$) which will replace an identical scalar field insertion in $W[C]$ which is removed by the scalar variation $\delta^A_{s,a}$ (or $\bar{\delta}^A_{s,a}$) acting directly on $W[C]$. Similarly, the fermion variations $\delta^{A,b}_s$ (or $\bar{\delta}^{A,b}_s$), when acting on the mass term of the fermion action, will bring down a factor of $\bar{\psi}_b[s]$ (or $\psi_b[s]$) which will replace an identical insertion in $W[C]$ removed by the second fermion variation $\bar{\delta}^{A,b}_s$ (or $\delta^{A,b}_s$) acting directly on $W[C]$. The net result, after using large N_c factorization, is

$$\begin{aligned}\langle \Delta\, W[C]\rangle = \frac{1}{2}\left(n_\ell + V'[\langle\chi\rangle]\, n_\mathrm{s} + m\, n_\mathrm{f}\right) \langle W[C]\rangle \,+ \\ + \langle \Delta\, W[C]\rangle_{\mathrm{deformation}} + \langle \Delta\, W[C]\rangle_{\mathrm{self-intersection}} + O\left(\frac{1}{N_c^2}\right), \end{aligned} \tag{3.17}$$

where n_ℓ is the number of (ordinary) links contained in the loop C, n_s is the number of scalar insertions (both ϕ's and $\phi^\dagger$'s), n_f is the number of fermion insertions (both ψ's and $\bar{\psi}$'s), and $\chi \equiv \frac{1}{N_c}\sum_a \mathrm{tr}(\phi^\dagger_a[s]\,\phi_a[s])$.[15]

As in the pure gauge theory, deformations of the loop C are produced by terms where one link variation acts on the pure gauge action and the other link variation acts on $W[C]$. In addition, there are now terms where one link variation acts on the hopping terms of the matter field action (and the other link variation acts on $W[C]$). Such terms have the effect of inserting two matter fields on either end of a link traversed by C. Finally, there are terms where either a scalar or fermion variation acts on the hopping terms of the matter field action (and a scalar or fermion variation acts on $W[C]$). These terms have the effect of moving a matter field insertion in $W[C]$ from its original site to some neighboring site. All these terms may be regarded as deformations of the initial loop C on the extended lattice $\bar{\Lambda}$ in which a plaquette is inserted into the loop C.[16] The result may be expressed as

$$\langle \Delta\, W[C]\rangle_{\mathrm{deformation}} = -\sum_{\ell\subset C}\;\sum_{p|\ell\subset\partial p} \frac{1}{4}\,\widetilde{\beta}_{\ell,p}\left\{\left\langle W[(\overline{\partial p})(\ell\bar{\ell})^{-1}C]\right\rangle + s_{\ell,p}\,\langle W[(\partial p)C]\rangle\right\}. \tag{3.18}$$

[15]The result (3.17) assumes that the lattice is translationally invariant, so that $\langle\chi[s]\rangle$ is independent of the site s. For such theories, note that the only dependence on the scalar potential $V[\chi]$ is via the single number $V'[\langle\chi\rangle]$; this is completely analogous to the large N_c universality of mixed adjoint-fundamental pure gauge actions [1, 2]. More generally, if the theory is not translationally invariant then $V'[\langle\chi\rangle]\,n_\mathrm{s}$ should be replaced by $V'[\langle\chi[s]\rangle]$ summed over those sites at which scalar insertions appear in $W[C]$.

[16]For deformations arising from derivatives of matter field links, it would be more accurate to say that some link ℓ is replaced by a "staple", i.e., the three sides of a plaquette other than ℓ. We will continue to refer to all such deformations as plaquette insertions — but the distinction is reflected in the presence of the $(\ell\bar{\ell})^{-1}$ factor in eq. (3.18).

The sum over plaquettes runs over all plaquettes in the extended lattice $\bar{\Lambda}$ whose boundary includes the link ℓ. Both C and ∂p are to be regarded as starting with link ℓ (oriented however it appears in C), so that the concatenation of ∂p with C makes sense. The factor of $(\ell\bar{\ell})^{-1}$ in the first deformed loop should be regarded as canceling the link ℓ which begins C and the link $\bar{\ell}$ which ends $\overline{\partial p}$. If directions of links are classified as 'gauge' (i.e., lying in the original lattice Λ), 'scalar', or 'fermion', then the plaquette weight $\widetilde{\beta}_{\ell,p}$ appearing in the result (3.18) is

$$\widetilde{\beta}_{\ell,p} \equiv \begin{cases} \dfrac{\beta_p}{N_c}\,, & \text{if } p \text{ is a `gauge-gauge' plaquette}\,; \\ \dfrac{\eta\kappa}{i}\,, & \text{if } p \text{ is a `gauge-fermion' plaquette}\,; \\ \kappa\,, & \text{if } p \text{ is a `gauge-scalar' plaquette}\,; \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.19}$$

For gauge-fermion plaquettes, the factor η appearing in eq. (3.19) is the fermion flavor connection $\eta[\ell]$ if the link ℓ being varied is a gauge link, but if ℓ is a fermion link, then $\eta = \eta[\ell']$ where ℓ' is the gauge link which precedes the forward directed fermion link in ∂p (i.e., the gauge link which runs from the ψ to the $\bar{\psi}$). The second term in the result (3.18) is present only for gauge links with unitary connections (for which a variation of $u[\ell]$ also varies $u[\bar{\ell}]$), and not for matter field links with non-unitary connections. For gauge links, the coefficient $s_{\ell,p}$ is just a $\pm$ sign,

$$s_{\ell,p} \equiv \begin{cases} +1, & \text{if } \ell \text{ is a gauge link and } p \text{ is a gauge-fermion plaquette}\,; \\ -1, & \text{if } \ell \text{ is a gauge link and } p \text{ is gauge-gauge or gauge-scalar}\,; \\ 0, & \text{if } \ell \text{ is a scalar or fermion link}\,. \end{cases} \tag{3.20}$$

Finally, the self-intersection terms involve a sum over all ways of breaking the loop C into two separate loops by reconnecting each distinct pair of traversals of any multiply-traversed link. But in the case of multiple traversals on matter field links, only pairs of traversals in opposite directions contribute (because ψ and $\bar{\psi}$, or ϕ and $\phi^\dagger$, are distinct). After using large N_c factorization, the result is[17]

$$\langle \Delta\, W[C] \rangle_{\rm self-intersection} = \sum_{\rm self-intersections} I[\ell]\, \langle W[C'] \rangle \, \langle W[C''] \rangle + O\left(\frac{1}{N_c^2}\right). \tag{3.21}$$

For parallel traversals of a gauge link ℓ, $C = C'C''$ with loops C, C' and C'' all regarded as starting with link ℓ. For antiparallel traversals, the loop C is to be regarded as $C = \ell C' \bar{\ell} C''$,

[17] Careful readers may note that double variations in Δ, when acting on loops with fermion insertions, can generate terms which do not correspond to geometric self-intersections of the loop C on the extended lattice $\bar{\Lambda}_{\rm f}$ and which are not present in the result (3.21). This is a consequence of our having assigned each distinct integration variable to two links in $\bar{\Lambda}_{\rm f}$, as indicated in eq. (3.10). However, the 'missing' terms correspond to splitting the original bosonic observable into a product of two fermionic observables. The expectation value of these terms, after using large N_c factorization, will always vanish.

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with ℓ positively oriented. The self-intersection coefficient $I[\ell]$ is

$$I[\ell] \equiv \begin{cases} +1, & \text{parallel traversals of a gauge link } \ell\,; \\ 0, & \text{parallel traversals of a scalar or fermion link } \ell\,; \\ -1, & \text{antiparallel traversals of link } \ell \text{ (of any type)}\,; \end{cases} \times$$
$$\times \begin{cases} +1, & \text{if link } \ell \text{ starts at a site in } \Lambda\,; \\ -1, & \text{if link } \ell \text{ starts at a site in } \widetilde{\Lambda}\,. \end{cases} \qquad (3.22)$$

Combining these pieces yields a loop equation for expectation values of single trace observables on the extended lattice $\bar{\Lambda}$ which closely resembles the result for a pure gauge theory,

$$\begin{aligned} \frac{1}{2}\left(n_\ell + V'[\langle\chi\rangle]\, n_{\rm s} + m\, n_{\rm f}\right) \langle W[C]\rangle = \sum_{\ell\subset C}\sum_{p|\ell\subset\partial p} \frac{1}{4}\,\widetilde{\beta}_{\ell,p}\Big[&\langle W[(\overline{\partial p})(\ell\bar{\ell})^{-1}C]\rangle + \\ &+ s_{\ell,p}\,\langle W[(\partial p)C]\rangle\Big] - \\ - \sum_{\rm self-intersections} \quad & I[\ell]\,\langle W[C']\rangle\,\langle W[C'']\rangle + \\ +O\left(\frac{1}{N_{\rm c}^2}\right). & \end{aligned} \qquad (3.23)$$

We will assume that $V'[\chi]$ is positive for non-negative arguments.[18] We will also assume that the fermion mass m is positive. As long as all fermion species have a common mass, this is only a matter of convention. Consequently, the coefficient of $\langle W[C]\rangle$ on the left side of the loop equation (3.23) is strictly positive. (The signs in the definition (3.15) of the operator Δ were chosen so that this would be true.) This means that one may iterate these loop equations to generate the strong coupling expansion (in the large $N_{\rm c}$ limit) for expectation values,[19] just like the pure gauge theory case. And, once again, this expansion is guaranteed to be convergent for sufficiently small values of $\widetilde{\beta}_{\ell,p}$. In light of (3.19), this means both small hopping parameter κ (equivalent to large scalar or fermion mass) as well as small $\beta_p/N_{\rm c}$ (or large 't Hooft coupling). So, at least in the strong coupling/large mass phase of the theory, the loop equations (3.23) completely determine the leading large $N_{\rm c}$ expectation values of single trace observables.

3.4 Multi-loop connected correlators

Applying exactly the same approach, the equations for two-loop correlators can be obtained from

$$\Big\langle \Delta\Big(\,(W[C_1] - \langle W[C_1]\rangle)\,(W[C_2] - \langle W[C_2]\rangle)\,\Big)\Big\rangle = 0\,. \qquad (3.24)$$

[18]The hopping term of the scalar action (3.3) differs from a lattice laplacian by a local term proportional to $\phi[s]^\dagger\phi[s]$ — this term has effectively been included in our scalar potential. Consequently, requiring positivity of $V'[\chi]$, even at $\chi = 0$, does not preclude the theory from being in a Higgs phase.

[19]Starting with vanishing expectation values for all observables except $\frac{1}{N_{\rm c}}\,{\rm tr}\,\mathbf{1}$, $\frac{1}{N_{\rm c}}\,{\rm tr}\,\bar{\psi}\psi$, and $\frac{1}{N_{\rm c}}\,{\rm tr}\,\phi^\dagger\phi$. The appropriate initial values for $\langle\bar{\psi}\psi\rangle$ and $\langle\phi^\dagger\phi\rangle$ follow from the loop equation (3.23) with all $\widetilde{\beta}_{\ell,p}$ set to zero. Specifically, $\langle\frac{1}{N_{\rm c}}\,{\rm tr}\,\bar{\psi}_b\psi_{b'}\rangle = \delta_{bb'}\,m^{-1} + O(\widetilde{\beta}_{\ell,p})$ and $\langle\frac{1}{N_{\rm c}}\,{\rm tr}\,\phi_a^\dagger\phi_{a'}\rangle = \delta_{aa'}\,\langle\chi\rangle_0/N_{\rm s} + O(\widetilde{\beta}_{\ell,p})$, with $\langle\chi\rangle_0$ the (positive) root of $V'[\langle\chi\rangle]\,\langle\chi\rangle = 1$.

This leads to the following equation for the connected two-loop correlators:

$$
\begin{aligned}
&\frac{1}{2}(n_\ell + V'[\langle\chi\rangle]\, n_{\rm s} + m\, n_{\rm f})\langle\langle W[C_1]W[C_2]\rangle\rangle \\
&= \Bigg[\sum_{\ell\subset C_1}\sum_{p|\ell\subset\partial p}\frac{1}{4}\,\widetilde{\beta}_{\ell,p}\Big[\langle\langle W[(\overline{\partial p})(\ell\bar{\ell})^{-1}C_1]\,W[C_2]\rangle\rangle + s_{\ell,p}\,\langle\langle W[(\partial p)C_1]\,W[C_2]\rangle\rangle\Big] \\
&\qquad + (C_1{\leftrightarrow}C_2)\Bigg] - \\
&\quad - \Bigg[\sum_{\substack{\rm self-intersections\\ (C_1)}} I[\ell]\,\Big[\langle\langle W[C_1']W[C_2]\rangle\rangle\,\langle W[C_1'']\rangle + \langle\langle W[C_1']W[C_2]\rangle\rangle\,\langle W[C_1'']\rangle\Big] + \\
&\qquad + (C_1{\leftrightarrow}C_2)\Bigg] - \\
&\quad - \sum_{\substack{\rm parallel\ gauge\\ \rm mutual\ intersections\\ (C_1,C_2)}} J[\ell]\,\langle W[C_1C_2]\rangle - \sum_{\substack{\rm anti-parallel\\ \rm mutual\ intersections\\ (C_1,C_2)}} K[\ell]\langle W[C_1(\ell\bar{\ell})^{-1}C_2]\rangle + \\
&\quad + \sum_{\substack{\rm parallel\ gauge\\ \rm mutual\ intersections\\ (\widetilde{C}_1,C_2)}} J[\ell]\,\langle W[\widetilde{C}_1C_2]\rangle + \sum_{\substack{\rm anti-parallel\\ \rm mutual\ intersections\\ (\widetilde{C}_1,C_2)}} K[\ell]\,\langle W[\widetilde{C}_1(\ell\bar{\ell})^{-1}C_2]\rangle + \\
&\quad + O\left(\frac{1}{N_{\rm c}^2}\right). \qquad (3.25)
\end{aligned}
$$

The numbers n_ℓ, $n_{\rm s}$, and $n_{\rm f}$ now denote the total numbers of links, scalar insertions, and fermion insertions (respectively) contained in both loops C_1 and C_2. For self-intersection terms involving loop C_1, if there is a parallel traversal of some link ℓ then $C_1 = C_1'C_1''$ with loops C_1, C_1' and C_1'' all regarded as starting with ℓ, while for self-intersections with antiparallel traversals, $C_1 = \ell C_1'\bar{\ell}C_1''$. (And likewise for self-intersections involving C_2.)

As noted in footnote 13, the Z_2 image $\widetilde{C}$ of any loop C represents the same observable, up to a minus sign, as does the loop C. In order to describe all joinings of the two loops which appear in the loop equation (3.24) as geometric intersections, one must consider mutual intersections between the two given loops in the extended lattice $\bar{\Lambda}$, as well as mutual intersections when either one of the loops is replaced by its Z_2 image (i.e., rigidly translated in a fermionic direction). This is why two sets of mutual intersection terms appear in the result (3.25). The 'parallel gauge mutual intersection' sums run over mutual intersections in which both loops traverse a gauge link (not a matter field link) in the same direction; both loops are to be regarded as starting with the intersection link ℓ.[20] The 'anti-parallel mutual intersection' sums run over mutual intersections in which loop C_2 traverses some link ℓ while loop C_1 (or $\widetilde{C}_1$) traverses $\bar{\ell}$; C_2 is to be regarded as starting

[20] If loops C_1 or C_2 multiply traverse a mutual intersection link ℓ, then each possible pairing of a traversal of ℓ in C_1 with a traversal of ℓ in C_2 generates a separate term in the parallel mutual intersection sum. Likewise, each possible pairing of a traversal of some link $\bar{\ell}$ in C_1 with a traversal of ℓ in C_2 generates a separate term in the antiparallel mutual intersection sum.

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with ℓ and C_1 (or $\widetilde{C}_1$) as ending with $\bar{\ell}$. The coefficient $J[\ell]$ for parallel traversals of a gauge link ℓ is

$$J[\ell] = \begin{cases} +1\,, & \text{if } \ell \in \Lambda; \\ -1\,, & \text{if } \ell \in \widetilde{\Lambda}, \end{cases} \tag{3.26}$$

while the coefficient $K[\ell]$ for antiparallel traversals of a link ℓ (of any type) is

$$K[\ell] = \begin{cases} -1, & \text{if } \ell \text{ is a gauge or scalar link}\,; \\ -1, & \text{if } \ell \text{ is a forward-directed fermion } (\bar{\psi})\,; \\ +1, & \text{if } \ell \text{ is a backward-directed fermion } (\psi)\,; \end{cases} \times$$
$$\times \begin{cases} +1, & \text{if link } \ell \text{ starts at a site in } \Lambda\,; \\ -1, & \text{if link } \ell \text{ starts at a site in } \widetilde{\Lambda}\,. \end{cases} \tag{3.27}$$

The form of the result (3.25) for connected correlators of decorated loops is completely analogous to the result (2.23) for two-loop correlators in pure gauge theory; the only differences are various fermionic minus signs and the absence of deformation and intersection terms associated with parallel traversals of matter field links. In the same manner discussed previously, these connected correlator loop equations may be solved iteratively (starting with all two-loop correlators equal to zero) to generate a strong coupling/large mass expansion with a non-zero radius of convergence. Consequently, these equations completely determine the leading large $N_{\rm c}$ limit of two-loop connected correlators (and hence the spectrum of particle masses[21]), at least in the strong coupling/large mass phase of the theory.

4. Orbifold theories

4.1 Orbifold projection

Start with a $\mathrm{U}(N_{\rm c})$ gauge theory of the form discussed in the last section [c.f. eqs. (3.2)–(3.4)], with $N_{\rm s}$ scalars and $N_{\rm f}$ fermions. This will be referred to as the "parent" theory. The global symmetry group of this theory is $G = \mathrm{U}(N_{\rm c}) \times \mathrm{U}(N_{\rm s}) \times \mathrm{U}(N_{\rm f})$, where the $\mathrm{U}(N_{\rm c})$ factor represents space-independent gauge transformations. To make an orbifold projection, one chooses a subgroup H of this global symmetry group and constructs a "daughter" theory by simply eliminating all degrees of freedom in the parent theory which are not invariant under the chosen subgroup H. (A similar explanation of how to construct daughter theories can be found in refs. [33, 34].)

We will only consider projections based on abelian subgroups, and will specifically focus on cases where

$$N_{\rm c} = k^d N\,, \tag{4.1}$$

for some positive integers k and d, and where H is a $(Z_k)^d$ subgroup of G chosen so that the subgroup of the $\mathrm{U}(k^d N)$ parent gauge group which commutes with H is $[\mathrm{U}(N)]^{k^d}$. This will

[21] Although our geometric encoding of observables is, at the moment, restricted to bosonic observables, the extension to fermionic observables, discussed in appendix A, is straightforward. The above assertion (that loop equations for connected correlators determine the particle spectrum) is valid for fermionic as well as bosonic channels.

be the gauge group of the daughter theory. To specify the desired $(Z_k)^d$ subgroup of G, it is sufficient to define the subgroup's d independent generators — call them η_α, $\alpha = 1, \cdots, d$. Each generator will be the product of some gauge transformation $\gamma_\alpha \in \mathrm{U}(N_\mathrm{c})$ times some non-gauge symmetry transformation $h_\alpha \in \mathrm{U}(N_\mathrm{s}) \times \mathrm{U}(N_\mathrm{f})$,

$$\eta_\alpha = \gamma_\alpha \times h_\alpha \,. \tag{4.2}$$

The gauge transformations $\{\gamma_\alpha\}$, regarded as $k^d N \times k^d N$ matrices, generate a representation of $(Z_k)^d$ and may be chosen to be

$$\gamma_\alpha = \underbrace{1_k \times \ldots}_{\alpha-1} \times \Omega \times \underbrace{1_k \times \ldots}_{d-\alpha} \times 1_N \,, \tag{4.3}$$

where 1_N and 1_k are $N \times N$ and $k \times k$ unit matrices, respectively, and

$$\Omega \equiv \mathrm{diag}(\omega^0, \omega^1, \ldots, \omega^{k-1}) \tag{4.4}$$

with $\omega \equiv e^{2\pi i/k}$. The factors $\{h_\alpha\}$ must be elements of a $\mathrm{U}(1)^{N_\mathrm{s}+N_\mathrm{f}}$ maximal abelian subgroup of the non-gauge $\mathrm{U}(N_\mathrm{s}) \times \mathrm{U}(N_\mathrm{f})$ symmetry group, and each must be a k'th root of unity. Hence one may write

$$h_\alpha = e^{2\pi i\, r_\alpha/k} \,, \tag{4.5}$$

where each r_α is a charge operator which assigns integer values to matter fields in the theory (and zero to all gauge links). (Different charge assignments will lead to differing daughter theories.)

If Φ denotes any variable (matter field or link variable) in the parent theory, all of which transform under the adjoint representation of the gauge group and hence may be regarded as a $k^d N \times k^d N$ matrices, then the action of the generator η_α on Φ is to transform

$$\Phi \mapsto e^{2\pi i r_\alpha(\Phi)/k}\, \gamma_\alpha \, \Phi \, \gamma_\alpha^{-1} \,, \tag{4.6}$$

where $r_\alpha(\Phi)$ is the value that the charge r_α assigns to the variable Φ. Consequently, the net effect of the orbifold projection is the imposition of the constraints

$$\Phi = e^{2\pi i r_\alpha(\Phi)/k}\, \gamma_\alpha \, \Phi \, \gamma_\alpha^{-1} \,, \qquad \alpha = 1, \ldots, d \,, \tag{4.7}$$

on each adjoint representation variable Φ.

At this point, it is useful to introduce the terminology of "theory space"[22] which provides the natural "habitat" for discussing the field content of the daughter theory. Theory space is a graph, denoted T, consisting of points and (directed) bonds.[23] Each point denotes a $\mathrm{U}(N)$ factor of the daughter theory gauge group. Each bond represents a matter field transforming under the fundamental representation of the gauge group factor at the originating end of the bond, and under the anti-fundamental representation of the

[22]The term "theory space" was introduced in [35]. Other often used names are "quiver diagrams" and "moose diagrams".

[23]We are avoiding use of the words "sites" and "links" to describe the theory space graph, to prevent confusion with the previous use of sites and links in reference to the spacetime lattice.

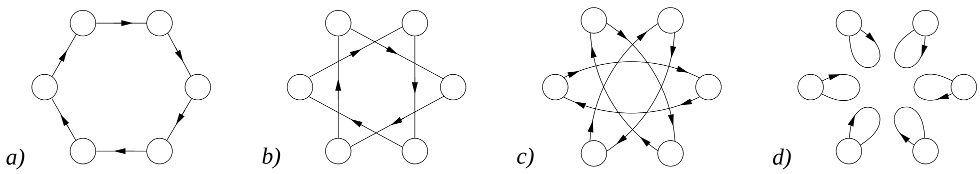

Figure 7: Theory space graphs obtained by applying a Z_6 orbifold projection to a $\mathrm{U}(6N)$ gauge theory containing a single adjoint representation fermion. The different graphs result from differing r charge assignments for the scalar field; graphs (a)–(d) correspond to $r = 1, 2, 3$ and 0, respectively. (Each bond represents a portion of the fermion field $\psi[s]$ which survives the orbifold projection; projections of $\bar\psi[s]$ correspond to reversing all arrows).

gauge group factor at the final end of the bond (and transforming as a singlet under all other gauge group factors); these are termed 'bifundamentals'. In our chosen case of a $(Z_k)^d$ orbifold, we have a theory space with k^d points which may be regarded as forming a regular, periodic lattice discretization of a d-dimensional torus. Theory space points [or associated $\mathrm{U}(N)$ factors of the daughter gauge group] may be labeled by a d-dimensional vector $\mathbf{j}$ whose components are integers running from 0 to $k-1$ (modulo k).

Let $\mathbf{r} = \{r_\alpha(\Phi)\}$ denote the vector of charge assignments for a particular field Φ. The link variables $u[\ell]$ must all have vanishing charge vectors, since they do not transform under the non-gauge symmetries h_α. Consequently, for link variables, the orbifold projection constraints (4.7) imply that each $k^dN \times k^dN$ unitary link matrix must be block-diagonal with k^d independent blocks, each of which is an $N \times N$ unitary matrix. Each block is the gauge connection, in the daughter theory, for one of the $\mathrm{U}(N)$ factors of the $\mathrm{U}(N)^{k^d}$ daughter gauge group; the individual blocks may be labeled as $u^{\mathbf{j}}[\ell]$ for $\mathbf{j} \in T$.

Each parent matter field, after the orbifold projection (4.7), generates k^d bifundamental fields in the daughter theory. For a matter field with charge vector $\mathbf{r}$, these bifundamental fields may be represented by bonds in the theory space connecting each point $\mathbf{j}$ with point $\mathbf{j}+\mathbf{r}$. More explicitly, the variables of the daughter theory are the unitary link variables $u^{\mathbf{j}}[\ell] \in \mathrm{U}(N)_{\mathbf{j}}$ belonging to each of the k^d gauge group factors, together with $N_{\mathrm{s}}\, k^d$ complex scalar bifundamentals $\phi^{\mathbf{j}}_a[s]$ and $N_{\mathrm{f}}\, k^d$ pairs of Grassmann bifundamentals $(\psi^{\mathbf{j}}_b[s], \bar\psi^{\mathbf{j}}_b[s])$ on each site of the (physical) lattice. The gauge transformation properties of the matter variables may be summarized as

$$\phi^{\mathbf{j}}_a[s]: \ \left(\square_{\mathbf{j}}, \overline{\square}_{\mathbf{j}+\mathbf{r}_a} \right), \tag{4.8}$$

$$\phi^{\mathbf{j}}_a[s]^\dagger: \ \left(\overline{\square}_{\mathbf{j}}, \square_{\mathbf{j}+\mathbf{r}_a} \right), \tag{4.9}$$

$$\psi^{\mathbf{j}}_b[s]: \ \left(\square_{\mathbf{j}}, \overline{\square}_{\mathbf{j}+\mathbf{r}^b} \right), \tag{4.10}$$

$$\bar\psi^{\mathbf{j}}_b[s]: \ \left(\overline{\square}_{\mathbf{j}}, \square_{\mathbf{j}+\mathbf{r}^b} \right), \tag{4.11}$$

where $\mathbf{r}_a$ $(a = 1, \ldots, N_{\mathrm{s}})$ is the vector of charge assignments for the parent scalar field $\phi_a[s]$, and $\mathbf{r}^b$ $(b = 1, \ldots, N_{\mathrm{f}})$ is the corresponding charge vector for the parent fermion field $\psi_b[s]$; note that these charge assignments are independent for each matter field.

Figure 7 illustrates the resulting theory space for Z_6 orbifold projections (i.e., $d = 1$ and $k = 6$) in a theory with one adjoint fermion and differing $\mathbf{r}$ charge assignments. These orbifold projections involve a single gauge transformation $\gamma \in \mathrm{U}(6N)$ [c.f. eq. (4.3)] which has the form

$$\gamma = \begin{pmatrix} 1_{N\times N} & & & \\ & \omega\, 1_{N\times N} & & \\ & & \ddots & \\ & & & \omega^5\, 1_{N\times N} \end{pmatrix} \tag{4.12}$$

where $\omega = e^{2\pi i/6}$. The link variables of the parent theory are $\mathrm{U}(6N)$ matrices. Since link variables have vanishing $\mathbf{r}$ charge, the effect of the projection (4.7) is to restrict each link variable $u[\ell]$ to be block-diagonal, with six blocks each of which is an $N\times N$ unitary matrix; these are precisely the daughter link variables $u^j[\ell]$, $j = 1, \cdots, 6$.

Since $d = 1$, the charge vector assigned to the fermion field $\psi[s]$ is only a single integer r. If the fermion is assigned vanishing charge, $r = 0$, then the orbifold projection restricts these variables to be block diagonal, just like the link variables. In this case, illustrated in figure 7d, the net effect is to reduce the $\mathrm{U}(6N)$ parent theory to six decoupled copies of a $\mathrm{U}(N)$ gauge theory with one adjoint fermion (i.e., the same theory as the parent except for the smaller gauge group).[24] If the charge r assigned to the fermion is non-vanishing, then the effect of the projection (4.7) is to restrict these variables to a form in which each variable has six $N \times N$ non-zero blocks that form a diagonal stripe displaced from the principle diagonal by $r \pmod 6$ steps.[25] As figure 7 illustrates, if $r = 1 \bmod 6$, one obtains a daughter theory with bifundamental fermions transforming under adjacent $\mathrm{U}(N)$ gauge group factors. There is a manifest Z_6 discrete symmetry which cyclically permutes the six gauge group factors. For $r = 2 \bmod 6$, one obtains two decoupled copies of a $\mathrm{U}(N)^3$ gauge theory in which a trio of bifundamental fermions connect the factors. For $r = 3 \bmod 6$, one has three decoupled $\mathrm{U}(N)^2$ gauge theories, each containing a pair of bifundamental fermions. The graphs for $r = 4$ or $5 \pmod 6$ are the same as those for $r = 2$ or 1, respectively, with the directions of arrows reversed; i.e., the daughter fermions are in conjugate representations.

Figure 8 illustrates the case of a $(Z_3)^2$ orbifold projection on a $\mathrm{U}(9N)$ gauge theory with one adjoint scalar and one adjoint fermion. We have chosen the $\mathbf{r}$ charges to be $(0, 1)$ for the scalar, and $(1, 0)$ for the fermion. All variables are now subjected to two constraints of the form (4.7). If each (adjoint representation) parent variable is divided into a 9×9 array of blocks, each of which is $N \times N$, then only 9 blocks from each variable will satisfy both constraints. The daughter theory has a $\mathrm{U}(N)^9$ gauge group, 9 bifundamental scalars, and 9 bifundamental fermions transforming as indicated in the theory space graph of figure 8. The graph should be regarded as periodic in both directions so that it is invariant under discrete translations. This reflects the fact that the daughter theory has a $(Z_3)^2$ discrete global symmetry which permutes the different gauge group factors.

[24]If one replaces the parent fermion by a scalar field then, in the daughter theory, the quartic self-interactions couple the six different scalars so that the daughter theory is no longer a product of six independent theories.

[25]If the parent field is divided into 36 blocks (each $N \times N$), labeled (j, j') with $j, j' = 1, \cdots, 6$, then the orbifold projection eliminates all blocks except those with $j' - j = r \bmod 6$.

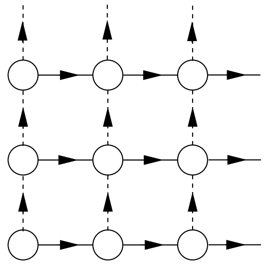

Figure 8: Theory space graph obtained from a $(Z_3)^2$ orbifold projection on a $\mathrm{U}(9N)$ gauge theory containing one adjoint scalar and one adjoint fermion. The $\mathbf{r}$ charge vector for the scalar is $(0,1)$ while that of the fermion is $(1,0)$. The graph is periodic in both directions; the dangling bonds at the top and right edges should be understood as wrapping around and connecting with the corresponding points along the bottom and left edges, respectively. Bonds drawn with solid lines represent bifundamental fermions, while dashed bonds represent bifundamental scalars.

Returning now to the discussion of our general class of $(Z_k)^d$ orbifolds, we will define the daughter theory action $S^{(\mathrm{d})}$ to be the result of replacing every variable in the parent theory action (3.2)–(3.4) by its orbifold projection, and then rescaling the action by a factor of $N/N_{\mathrm{c}} = k^{-d}$. Including this rescaling will be necessary to make the daughter theory loop equations isomorphic to those of the parent theory. The resulting action of the daughter theory is

$$S^{(\mathrm{d})} = S^{(\mathrm{d})}_{\mathrm{gauge}} + S^{(\mathrm{d})}_{\mathrm{scalar}} + S^{(\mathrm{d})}_{\mathrm{fermion}}\,, \tag{4.13}$$

with

$$S^{(\mathrm{d})}_{\mathrm{gauge}} \equiv \sum_{\mathbf{j}\in T} {\sum_{p\in\Lambda}}' \beta^{(\mathrm{d})}_p \; \mathrm{Re}\,\mathrm{tr}\, u^{\mathbf{j}}[\partial p]\,, \tag{4.14}$$

and

$$\frac{\beta^{(\mathrm{d})}_p}{N} \equiv \frac{\beta_p}{N_{\mathrm{c}}}\,. \tag{4.15}$$

Note that this gauge action involves a sum over each point in the theory space T (i.e., a sum over each $\mathrm{U}(N)$ gauge group factor). The condition (4.15) is equivalent to the requirement that the 't Hooft couplings (g^2N) coincide in the parent and daughter theories. The scalar action in the daughter theory is

$$\begin{aligned} S^{(\mathrm{d})}_{\mathrm{scalar}} = N\Bigg\{ &\sum_{\ell=\langle ss'\rangle\in\Lambda} \sum_{a=1}^{N_{\mathrm{s}}} \sum_{\mathbf{j}\in T} \frac{1}{2}\,\kappa\; \mathrm{tr}\Big(\phi^{\mathbf{j}}_a[s]^\dagger u^{\mathbf{j}}[\ell]\,\phi^{\mathbf{j}}_a[s']\,u^{\mathbf{j}+\mathbf{r}_a}[\bar\ell]\Big) - \\ &- \sum_{s\in\Lambda} N_{\mathrm{c}}\, V\Bigg[\sum_{a=1}^{N_{\mathrm{s}}} \sum_{\mathbf{j}\in T} \frac{\mathrm{tr}}{N_{\mathrm{c}}}\Big(\phi^{\mathbf{j}}_a[s]^\dagger \phi^{\mathbf{j}}_a[s]\Big)\Bigg]\Bigg\}\,. \end{aligned} \tag{4.16}$$

This generalization of the parent scalar action (3.3) describes a set of k^d scalars which are in either adjoint or bifundamental representations, depending on whether their $\mathbf{r}$ charges vanish or are non-zero. Finally, the fermion action in the daughter theory is

$$S^{(\mathrm{d})}_{\mathrm{fermion}} = N \sum_{b=1}^{N_{\mathrm{f}}} \sum_{\mathbf{j}\in T} \Bigg\{ \sum_{\ell=\langle ss'\rangle\in\Lambda} \frac{1}{2i}\,\kappa\; \mathrm{tr}\Big(\bar\psi^{\mathbf{j}}_b[s]\,\eta[\ell]\,u^{\mathbf{j}}[\ell]\,\psi^{\mathbf{j}}_b[s']\,u^{\mathbf{j}+\mathbf{r}^b}[\bar\ell]\Big) -$$

$$- \sum_{s \in \Lambda} m \ \mathrm{tr}\Big(\bar{\psi}^{\mathbf{j}}_{b}[s]\, \psi^{\mathbf{j}}_{b}[s]\Big) \Bigg\} . \tag{4.17}$$

In addition to whatever discrete translation and rotational symmetries are possessed by the euclidean lattice Λ, the daughter theory action (4.13), and associated integration measure, are invariant under independent $\mathrm{U}(N)$ gauge transformations in each of the k^d gauge group factors. The daughter theory is also invariant under a $(Z_k)^d$ global symmetry which permutes the different gauge group factors and fields of the daughter theory in the manner dictated by the discrete translation symmetry of the periodic theory space graph T. Finally, the daughter theory is invariant under whatever subgroup of the global $\mathrm{U}(N_{\mathrm{s}}) \times \mathrm{U}(N_{\mathrm{f}})$ flavor rotation group commutes with the non-gauge transformations $\{h_\alpha\}$ used to define the orbifold projection.[26]

As already illustrated by the daughter theory action (4.13), the orbifold projection connecting parent and daughter theories generates a natural mapping between observables of the parent theory and the subclass of observables of the daughter theory which are invariant under the $(Z_k)^d$ global symmetry. To see this explicitly, consider first an ordinary Wilson loop $W[C]$ under, for simplicity, a Z_k orbifold projection. If the link variables composing the Wilson loop in the parent theory are replaced by their orbifold projections, then the Wilson loop operator becomes an average of loop operators (for the same contour) in each of the k different $\mathrm{U}(N)$ gauge group factors of the daughter theory,

$$W[C] \equiv \frac{1}{kN} \ \mathrm{tr}\ u[C] \longrightarrow \frac{1}{k} \sum_{j=1}^{k} \frac{1}{N} \ \mathrm{tr}\ u^j[C] \equiv W_{\mathrm{d}}[C] \,. \tag{4.18}$$

The trace on the left side of the map involves $kN \times kN$ matrices, while that on the right involves $N \times N$ matrices. As indicated here, we will use $W_{\mathrm{d}}[C]$ to denote daughter theory Wilson loops which are averaged over daughter theory gauge group factors (or equivalently, over all points of the theory space T).

Now consider of a decorated Wilson loop (i.e., one with matter field insertions) such as, for example, $W[\Gamma_1, \Gamma_2]_{b\bar{b}} \equiv \frac{1}{kN}\ \mathrm{tr}\ \psi_b[s_1] u[\Gamma_1] \bar{\psi}_b[s_2] u[\Gamma_2]$. If all variables are replaced by their orbifold projections, then

$$\begin{aligned} W[\Gamma_1, \Gamma_2]_{b\bar{b}} &\equiv \frac{1}{kN}\, \mathrm{tr}\, \psi_b[s_1]\, u[\Gamma_1]\, \bar{\psi}_b[s_2]\, u[\Gamma_2] \\ &\downarrow \\ W_{\mathrm{d}}[\Gamma_1, \Gamma_2]_{b\bar{b}} &\equiv \frac{1}{k} \sum_{j=1}^{k} \frac{1}{N} \ \mathrm{tr}\ \psi^j_b[s_1]\, u^{j+r^b}[\Gamma_1]\, \bar{\psi}^j_b[s_2]\, u^j[\Gamma_2] \,, \end{aligned} \tag{4.19}$$

where r^b is the $\mathbf{r}$ charge assigned to fermion ψ_b. If the charge r^b is non-zero then the daughter fermion ψ^j_b is in a bifundamental representation, which means it transforms under

[26]Retaining the full $\mathrm{U}(N_{\mathrm{s}})$ symmetry in the daughter theory requires that all scalars have the same $\mathbf{r}$ charge. Retaining $\mathrm{U}(N_{\mathrm{f}})$ symmetry likewise requires that all fermions have a common $\mathbf{r}$ charge. If distinct $\mathbf{r}$ charges are assigned to different flavors of fermions or scalars, then the global flavor symmetry of the daughter theory will be a smaller subgroup of $\mathrm{U}(N_{\mathrm{s}}) \times \mathrm{U}(N_{\mathrm{f}})$.

different gauge group factors on the left and right. This is why the appropriate gauge connections (emerging directly from the orbifold projection) involve u^j when acting on the left of ψ_b^j, and u^{j+r^b} when acting on the right. The result is a gauge invariant operator in the daughter theory (as it must be) which, once again, is averaged over all "starting points" in theory space and thus is invariant under the Z_k global symmetry of theory space.

More generally, if $\mathcal{O}$ is any operator of the parent theory which is both gauge invariant and invariant under the non-gauge symmetry transformations $\{h_a\}$ used in defining the orbifold projection, then the projection will map this operator into an operator $\mathcal{O}_{\rm d}$ in the daughter theory which is both gauge invariant and invariant under the global $(Z_k)^d$ translation symmetry of theory space.[27] As a final example, consider a Wilson loop decorated by any number of fermion (or antifermion) insertions in a multi-flavor theory. Under a general $(Z_k)^d$ orbifold projection,

$$\begin{aligned}
W[\Gamma_1,\Gamma_2,\ldots,\Gamma_K]_{b_1\bar{b}_2\cdots b_K} &= \frac{1}{k^d N}\,\mathrm{tr}\left(\psi_{b_1}[s_1]u[\Gamma_1]\bar{\psi}_{b_2}[s_2]\,u[\Gamma_2]\cdots u[\Gamma_{K-1}]\psi_{b_K}[s_K]\,u[\Gamma_K]\right)\\
&\downarrow\\
W_{\rm d}[\Gamma_1,\Gamma_2,\ldots,\Gamma_K]_{b_1\bar{b}_2\cdots b_K} &\equiv \frac{1}{k^d}\sum_{\mathbf{j}\in T}\frac{1}{N}\,\mathrm{tr}\left(\psi^{\mathbf{j}}_{b_1}[s_1]\,u^{\mathbf{j}+\mathbf{r}^{b_1}}[\Gamma_1]\,\bar{\psi}^{\mathbf{j}+\mathbf{r}^{b_1}}_{b_2}[s_2]\,u^{\mathbf{j}+\mathbf{r}^{b_1}-\mathbf{r}^{b_2}}[\Gamma_2]\cdots\right.\\
&\qquad\left.\cdots u^{\mathbf{j}-\mathbf{r}^{b_K}}[\Gamma_{K-1}]\,\psi^{\mathbf{j}-\mathbf{r}^{b_K}}_{b_K}[s_K]\,u^{\mathbf{j}}[\Gamma_K]\right), \qquad (4.20)
\end{aligned}$$

provided the sum of $\mathbf{r}$ charges of all the fermion insertions vanish, $\mathbf{r}^{b_1}-\mathbf{r}^{b_2}+\cdots+\mathbf{r}^{b_K}=0$ (mod k). (Otherwise, the operator maps to zero under the orbifold projection.) Associating each variable with a point or bond in theory space, as discussed earlier, this condition is the same as the requirement that the path *in theory space* traversed by a single-trace operator must be closed. Note that given an arbitrary starting point $\mathbf{j}$ in theory space, the transformation properties of each variable in the parent operator uniquely determine the path in theory space associated with the daughter operator. The starting point $\mathbf{j}$ is averaged over all points in theory space, thereby explicitly constructing a $(Z_k)^d$ invariant result.

Once again, instead of displaying explicitly the path segments and insertions in decorated loops [as in eqs. (4.19) and (4.20)], one may instead associate every such decorated loop with a closed contour C in the extended lattice $\bar{\Lambda}$,[28] and write the operator mapping of arbitrary (bosonic) single-trace operators in the trivial form

$$W[C] \longrightarrow W_{\rm d}[C]\,, \qquad C\in\bar{\Lambda}\,. \qquad (4.21)$$

The essential point is that any closed path C in the extended lattice uniquely identifies both the associated operator $W[C]$ in the parent theory, and the corresponding $(Z_k)^d$ invariant operator $W_{\rm d}[C]$ in the daughter theory.

[27] If the operator $\mathcal{O}$ transforms non-trivially (and irreducibly) under the $\{h_a\}$ non-gauge symmetries, then it maps to zero under the orbifold projection. A simple example is tr ϕ^n in a Z_k orbifold with a scalar field ϕ. If ϕ has non-zero $\mathbf{r}$ charge, then after the orbifold projection ϕ^n will be block off-diagonal and tr ϕ^n will vanish unless $n\,r$ is divisible by k. The condition that $\mathcal{O}$ be invariant under the $\{h_a\}$ non-gauge symmetries amounts to the requirement that the $\mathbf{r}$-charges of all matter field insertions sum to zero.

[28] As stated earlier, we are assuming for the moment that all operators are bosonic.

4.2 Loop equations in daughter orbifold theories

The previous treatment of loop equations in theories with adjoint matter fields may be generalized to daughter orbifold theories in a straightforward fashion. The operator Δ [c.f., eq. (3.15)] which generated our previous loop equations must merely be redefined to include a sum over all points in theory space,

$$\Delta \equiv \frac{1}{N}\, e^{-S^{(\mathrm{d})}} \sum_{\mathbf{j}\in T} \Bigg\{ {\sum_{\ell\in\Lambda}}' \, \delta_\ell^{A,\mathbf{j}}\, e^{S^{(\mathrm{d})}}\, \delta_\ell^{A,\mathbf{j}} \sum_{s\in\Lambda} \Big[\bar\delta_{s,a}^{A,\mathbf{j}}\, e^{S^{(\mathrm{d})}}\, \delta_{s,a}^{A,\mathbf{j}} + \delta_{s,a}^{A,\mathbf{j}}\, e^{S^{(\mathrm{d})}}\, \bar\delta_{s,a}^{A,\mathbf{j}} + \\ + \bar\delta_{s}^{A,\mathbf{j},b}\, e^{S^{(\mathrm{d})}}\, \delta_{s}^{A,\mathbf{j},b} - \delta_{s}^{A,\mathbf{j},b}\, e^{S^{(\mathrm{d})}}\, \bar\delta_{s}^{A,\mathbf{j},b} \Big] \Bigg\} . \tag{4.22}$$

Here $\delta_\ell^{A,\mathbf{j}}$ is the link variation previously defined in eq. (2.7), but now acting specifically on the link variable $u^{\mathbf{j}}[\ell]$. Similarly, $\delta_{s,a}^{A,\mathbf{j}}$ is a scalar variation as defined in eq. (3.13) but now acting on $\phi_a^{\mathbf{j}}[s]$, and $\delta_s^{A,\mathbf{j},b}$ is the fermion variation as defined in eq. (3.14) but now acting on $\psi_b^{\mathbf{j}}[s]$, *etc.* The integral of any variation still vanishes, so the loop equation for any observable $\mathcal{O}$ in the daughter theory may once again be written as

$$0 = \langle \Delta\, \mathcal{O} \rangle . \tag{4.23}$$

For any closed contour C in the extended lattice $\bar\Lambda$, let $W[C]$ denote the associated single-trace decorated Wilson loop in the parent theory, and $W_{\mathrm{d}}[C]$ the corresponding single-trace $(Z_k)^d$ invariant decorated Wilson loop in the daughter theory. Just as in the parent theory, the daughter theory loop equation for $W_{\mathrm{d}}[C]$ will involve a sum of three types of terms: terms proportional to $\langle W_{\mathrm{d}}[C]\rangle$, terms involving single plaquette deformations (in the extended lattice) of the contour C, and self-intersection terms.

Terms proportional to $\langle W_{\mathrm{d}}[C]\rangle$ are generated when both link variations $\delta_\ell^{A,\mathbf{j}}$ in the operator Δ act on the same link variable $u^{\mathbf{j}}[\ell]$ present in (some piece of) $W_{\mathrm{d}}[C]$. Such terms are also generated when one matter field variation acts on the local part of the matter field action (the fermion mass term or the scalar potential term) and the other variation acts on an insertion of the conjugate matter field in $W_{\mathrm{d}}[C]$. Due to the inclusion of a sum over all theory space points in the definition (4.22), the resulting contribution from each gauge link, scalar, or fermion insertion in $W_{\mathrm{d}}[C]$ is independent of the theory space index on the variable. The rescaling of the fermion mass and scalar potential terms in the daughter theory action (relative to the parent action) by a factor of N/N_{c} is exactly what is needed so that, by construction, the resulting coefficient of $\langle W_{\mathrm{d}}[C]\rangle$ is the same as in the parent theory. In other words,

$$\langle \Delta\, W_{\mathrm{d}}[C]\rangle = \frac{1}{2}\left(n_\ell + V'[\langle\chi\rangle]\, n_{\mathrm{s}} + m\, n_{\mathrm{f}}\right) \langle W_{\mathrm{d}}[C]\rangle + \\ + \langle \Delta\, W_{\mathrm{d}}[C]\rangle_{\mathrm{deformation}} + \langle \Delta\, W_{\mathrm{d}}[C]\rangle_{\mathrm{self-intersection}} + O\left(\frac{1}{N^2}\right), \tag{4.24}$$

where $\chi \equiv \sum_{\mathbf{j},a} \frac{\mathrm{tr}}{N_{\mathrm{c}}}(\phi_a^{\mathbf{j}}[s]^\dagger\, \phi_a^{\mathbf{j}}[s])$, and n_ℓ denotes the number of gauge links in the contour C, n_{s} the total number of scalar insertions, and n_{f} the total number of fermion insertions.

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Deformations of the loop C are produced whenever one link variation $\delta_\ell^{a,\mathbf{j}}$ acts on the gauge action $S^{(\mathrm{d})}_{\mathrm{gauge}}$ and the other variation acts on a gauge link $u^{\mathbf{j}}[\ell]$ present in $W_{\mathrm{d}}[C]$, or when a matter field variation acts on the hopping terms in the action and the conjugate variation acts on a matter field insertion in $W_{\mathrm{d}}[C]$. As described earlier, all of these terms may be regarded as plaquette deformations in the extended lattice $\bar{\Lambda}$; the fact that all variables now carry an additional theory space label $\mathbf{j}$ makes no difference. The result may be written in the form

$$\langle \Delta\, W_{\mathrm{d}}[C]\rangle_{\mathrm{deformation}} = -\sum_{\ell\subset C}\sum_{p|\ell\subset\partial p} \frac{1}{4}\,\widetilde{\beta}^{(\mathrm{d})}_{\ell,p}\,\{\langle W_{\mathrm{d}}[(\overline{\partial p})(\ell\bar{\ell})^{-1}C]\rangle + s_{\ell,p}\,\langle W_{\mathrm{d}}[(\partial p)C]\rangle\}\,, \tag{4.25}$$

where the coefficient $\widetilde{\beta}^{(\mathrm{d})}_{\ell,p}$ equals $\beta^{(\mathrm{d})}_p/N$ when p is a 'gauge-gauge' plaquette, and is otherwise the same as $\widetilde{\beta}_{\ell,p}$ as defined in eq. (3.19). [And $s_{\ell,p}$ is the same coefficient defined previously in eq. (3.20).] Hence, given the relation (4.15) between parent and daughter gauge couplings, this result coincides precisely with the corresponding deformation term (3.18) in the parent theory.

The final contributions to the loop equation for $W_{\mathrm{d}}[C]$ are self-intersection terms produced by double variations in Δ acting on multiply traversed links. In these terms, there *is* a potential difference between parent and daughter theories. When a decorated Wilson loop is represented as a closed contour in the extended lattice, self-intersection terms in the parent theory loop equation may be regarded as geometric; every pair of traversals of any given link (in opposite directions for matter field links, and either direction for gauge links) generates a self-intersection contribution. In the daughter theory, analogous self-intersection terms are only present when the two traversals of the given link represent variables with the same theory space label $\mathbf{j}$. This follows directly from the structure of the operator Δ (4.22): gauge invariance dictates that both variations in each term act on variables at the same place in theory space.

As a concrete example, consider the parent theory observable

$$\mathcal{O} \equiv \frac{\mathrm{tr}}{N_{\mathrm{c}}}\left(\phi_a[s]\,u[C_1]\,\phi_a[s]\,u[C_2]\,\phi_a[s]^\dagger\,u[C_3]\,\phi_a[s]^\dagger\,u[C_4]\right), \tag{4.26}$$

containing four scalar field insertions all at the same site s. ($C_1,\cdots,C_4$ are all closed loops in the physical lattice Λ which begin at site s.) The corresponding daughter theory observable is

$$\mathcal{O}_{\mathrm{d}} \equiv \frac{1}{k^d}\sum_{\mathbf{j}\in T}\frac{\mathrm{tr}}{N}\left(\phi_a^{\mathbf{j}}[s]\,u^{\mathbf{j}+\mathbf{r}_a}[C_1]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]\,u^{\mathbf{j}+2\mathbf{r}_a}[C_2]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]^\dagger\,u^{\mathbf{j}+\mathbf{r}_a}[C_3]\,\phi_a^{\mathbf{j}}[s]^\dagger\,u^{\mathbf{j}}[C_4]\right). \tag{4.27}$$

In the loop equation for $\mathcal{O}$, the self-intersection terms (after using large N_{c} factorization) generated by double variations of the scalar fields are

$$\begin{aligned}\langle\mathcal{O}\rangle_{\mathrm{self-intersection}} &= -\left\langle\frac{\mathrm{tr}}{N_{\mathrm{c}}}\,u[C_1]\,\phi_a[s]\,u[C_2]\,\phi_a[s]^\dagger\,u[C_3]\right\rangle\left\langle\frac{\mathrm{tr}}{N_{\mathrm{c}}}\,u[C_4]\right\rangle - \\ &\quad -\left\langle\frac{\mathrm{tr}}{N_{\mathrm{c}}}\,u[C_1]\,\phi_a[s]\,u[C_2]\right\rangle\left\langle\frac{\mathrm{tr}}{N_{\mathrm{c}}}\,u[C_3]\,\phi_a[s]^\dagger\,u[C_4]\right\rangle -\end{aligned}$$

$$-\left\langle\frac{\operatorname{tr}}{N_{\rm c}}\,u[C_2]\,\phi_a[s]^\dagger\,u[C_3]\right\rangle\left\langle\frac{\operatorname{tr}}{N_{\rm c}}\,u[C_4]\,\phi_a[s]\,u[C_1]\right\rangle-$$
$$-\left\langle\frac{\operatorname{tr}}{N_{\rm c}}\,u[C_2]\right\rangle\left\langle\frac{\operatorname{tr}}{N_{\rm c}}\,u[C_3]\,\phi_a[s]^\dagger\,u[C_4]\,\phi_a[s]\,u[C_1]\right\rangle. \qquad (4.28)$$

In the daughter theory loop equation for $\mathcal{O}_{\rm d}$, the analogous self-intersection terms coming from double variations of $\phi_a^{\mathbf{j}}[s]$ or $\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]$ are

$$\begin{aligned}\langle\mathcal{O}_{\rm d}\rangle_{\rm self-intersection} &= -\frac{1}{k^d}\sum_{\mathbf{j}\in T}\left\langle\frac{\operatorname{tr}}{N}u^{\mathbf{j}+\mathbf{r}_a}[C_1]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]\,u^{\mathbf{j}+2\mathbf{r}_a}[C_2]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]^\dagger\,u^{\mathbf{j}+\mathbf{r}_a}[C_3]\right\rangle\times\\ &\quad\times\left\langle\frac{\operatorname{tr}}{N}u^{\mathbf{j}}[C_4]\right\rangle-\frac{1}{k^d}\sum_{\mathbf{j}\in T}\left\langle\frac{\operatorname{tr}}{N}\,u^{\mathbf{j}+2\mathbf{r}_a}[C_2]\right\rangle\times\\ &\quad\times\left\langle\frac{\operatorname{tr}}{N}\,u^{\mathbf{j}+\mathbf{r}_a}[C_3]\,\phi_a^{\mathbf{j}}[s]^\dagger\,u^{\mathbf{j}}[C_4]\,\phi_a^{\mathbf{j}}[s]\,u^{\mathbf{j}+\mathbf{r}_a}[C_1]\right\rangle, \qquad (4.29)\end{aligned}$$

assuming that $\mathbf{r}_a\neq 0$.[29]

Comparing the parent and daughter results, there are two sources of "mismatch". First and foremost, the two intersection terms in the daughter theory result (4.29) resemble the first and last terms in the parent theory result (4.28), but terms corresponding to the second and third terms of the parent theory result are completely absent. Second, under the parent/daughter operator mapping every single-trace parent observable maps into a sum over theory space of single-trace daughter observables. Hence, each product of expectation values in the parent self-intersection terms (4.28) should map onto a product of independent sums over theory space of single expectation values. Instead, the daughter theory intersection terms (4.29) involve a single sum over theory space of a product of expectation values.

Dealing with the second point first, note that the discrete $(Z_k)^d$ symmetry of theory space guarantees that the expectation value of any daughter theory operator involving terms at particular points in theory space will coincide with the average of the expectation value over all points in theory space — *provided the* $(Z_k)^d$ *theory space symmetry is not spontaneously broken in the daughter theory.* Consequently, if the daughter theory is in a phase with unbroken $(Z_k)^d$ symmetry, then

$$\begin{aligned}&\frac{1}{k^d}\sum_{\mathbf{j}\in T}\left\langle\frac{\operatorname{tr}}{N}\,u^{\mathbf{j}+\mathbf{r}_a}[C_1]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]\,u^{\mathbf{j}+2\mathbf{r}_a}[C_2]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]^\dagger\,u^{\mathbf{j}+\mathbf{r}_a}[C_3]\right\rangle\left\langle\frac{\operatorname{tr}}{N}\,u^{\mathbf{j}}[C_4]\right\rangle=\\ &=\left[\frac{1}{k^d}\sum_{\mathbf{j}\in T}\left\langle\frac{\operatorname{tr}}{N}\,u^{\mathbf{j}+\mathbf{r}_a}[C_1]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]\,u^{\mathbf{j}+2\mathbf{r}_a}[C_2]\,\phi_a^{\mathbf{j}+\mathbf{r}_a}[s]^\dagger\,u^{\mathbf{j}+\mathbf{r}_a}[C_3]\right\rangle\right]\times\\ &\quad\times\left[\frac{1}{k^d}\sum_{\mathbf{j}'\in T}\left\langle\frac{\operatorname{tr}}{N}\,u^{\mathbf{j}'}[C_4]\right\rangle\right], \qquad (4.30)\end{aligned}$$

etc.

To address the "missing" analogues of the second and third terms in the parent theory result (4.28), note that these terms involve expectation values of operators, such as

[29]If $\mathbf{r}_a=0$, then two additional self-intersection terms are generated which resemble the second and third terms in the parent theory result (4.28), namely $-\sum_{\mathbf{j}}\langle\frac{\operatorname{tr}}{N}u^{\mathbf{j}}[C_1]\phi_a^{\mathbf{j}}[s]u^{\mathbf{j}}[C_2]\rangle\langle\frac{\operatorname{tr}}{N}u^{\mathbf{j}}[C_3]\phi_a^{\mathbf{j}}[s]^\dagger u^{\mathbf{j}}[C_4]\rangle-\sum_{\mathbf{j}}\langle\frac{\operatorname{tr}}{N}u^{\mathbf{j}}[C_2]\phi_a^{\mathbf{j}}[s]^\dagger u^{\mathbf{j}}[C_3]\rangle\langle\frac{\operatorname{tr}}{N}u^{\mathbf{j}}[C_4]\phi_a^{\mathbf{j}}[s]u^{\mathbf{j}}[C_1]\rangle$.

$\mathrm{tr}(\phi_a[s]\, u[C_2 C_1])$, which are gauge invariant but are not invariant under U(1) phase rotations of the scalar field. More specifically, the expectations in these terms are not invariant under the Z_k transformations h_α used to define the orbifold projection, under which ϕ_a acquires a phase of $e^{2\pi i r_\alpha/k}$. (Unless $\mathbf{r}_a = 0$, in which case these terms are invariant and, as noted in footnote 29, analogues of these terms do then appear in the daughter theory result.) These Z_k phase rotations are symmetries of the parent theory, and these symmetries guarantee that expectation values of operators transforming non-trivially (and irreducibly) under these symmetries will vanish — *provided the non-gauge symmetries used to define the orbifold projection are not spontaneously broken in the parent theory.*[30]

To recap, the second and third terms in the parent theory self-intersection result (4.28) will vanish, and the first and last terms will match the daughter theory self-intersection result (4.29), provided the parent theory is in a phase which respects the non-gauge symmetries used to construct the orbifold projection and the daughter theory is in a phase which respects the $(Z_k)^d$ translation symmetry of theory space. Although these points have been illustrated with the particular example of the operator (4.26), the conclusion is general: the self-intersection terms in the loop equation of any single-trace observable coincide in the parent and daughter theories (under the parent/daughter operator mapping) provided the orbifold and theory space symmetries are unbroken in the parent and daughter theories, respectively.

The net result, under the assumption of appropriate unbroken symmetries, is that the loop equations for arbitrary single-trace observables in the daughter theory have exactly the same form as in the parent theory, namely

$$\begin{aligned} \frac{1}{2}\left(n_\ell + V'[\langle\chi\rangle]\, n_{\rm s} + m\, n_{\rm f}\right)\langle W_{\rm d}[C]\rangle = {} & \sum_{\ell\subset C}\sum_{p|\ell\subset\partial p} \frac{1}{4}\widetilde{\beta}^{(\rm d)}_{\ell,p}\Big\{\langle W_{\rm d}[(\overline{\partial p})(\ell\bar{\ell})^{-1}C]\rangle + \\ & \qquad\qquad + s_{\ell,p}\,\langle W_{\rm d}[(\partial p)C]\rangle\Big\} - \\ & - \sum_{\rm self-intersections} I[\ell]\;\langle W_{\rm d}[C']\rangle\,\langle W_{\rm d}[C'']\rangle + \\ & + O\left(\frac{1}{N^2}\right), \end{aligned} \tag{4.31}$$

where the splitting coefficient $I[\ell]$ is defined in eq. (3.22) and, as before, $C = C'C''$ with loops C, C' and C'' all regarded as starting with link ℓ for parallel traversals of the intersection link, while $C = \ell C'\bar{\ell}C''$ for antiparallel traversals.[31]

[30] Some readers may wonder whether these symmetry realization restrictions are necessary, since all symmetry non-invariant operators will have vanishing expectation values in a lattice measure containing no symmetry-breaking boundary conditions or perturbations — regardless of the phase of the theory. Recall, however, that in the absence of symmetry breaking perturbations, spontaneous symmetry breaking is signaled by the breakdown of cluster decomposition in correlators of symmetry violating order parameters. Large $N_{\rm c}$ factorization, which we have used in deriving our loop equations, holds only in states which satisfy cluster decomposition. Therefore, in any phase of the theory which has spontaneous symmetry breaking, the lattice measure should tacitly be understood to include some perturbation which picks out a preferred equilibrium state satisfying cluster decomposition.

[31] This geometric description of the self-intersection terms includes those terms which have just been

Implications of the equality of the above loop equations between parent and daughter theories (for single trace observables, suitably mapped between the two theories) will be discussed in the next section.

4.3 Multi-loop connected correlators

Extending the derivation of loop equations to multi-loop connected correlators is straightforward, and proceeds in complete analogy with the parent theory treatment. We only briefly sketch the two loop case. Loop equations for two-loop connected correlators in the daughter theory are generated by the identity

$$\Big\langle \Delta \Big(\left(W_{\rm d}[C_1] - \langle W_{\rm d}[C_1]\rangle\right) \left(W_{\rm d}[C_2] - \langle W_{\rm d}[C_2]\rangle\right) \Big)\Big\rangle = 0\,, \tag{4.32}$$

with Δ given in eq. (4.22). Evaluating this in the same fashion described previously, using large $N_{\rm c}$ factorization plus unbroken $(Z_k)^d$ translation invariance in theory space, yields a loop equation which may be written in exactly the same form as the previous parent theory result (assuming unbroken orbifold projection symmetries in the parent), namely

$$\begin{aligned}
&\frac{1}{2}(n_\ell + V'[\langle\chi\rangle]\, n_{\rm s} + m\, n_{\rm f})\langle\langle W_{\rm d}[C_1] W_{\rm d}[C_2]\rangle\rangle = \\
&= \Bigg[\sum_{\ell\subset C_1}\sum_{p|\ell\subset\partial p} \frac{1}{4}\,\widetilde{\beta}_{\ell,p}\Big[\langle\langle W_{\rm d}[(\overline{\partial p})(\ell\bar{\ell})^{-1}C_1]\, W_{\rm d}[C_2]\rangle\rangle + s_{\ell,p}\,\langle\langle W_{\rm d}[(\partial p)C_1]\, W_{\rm d}[C_2]\rangle\rangle\Big] - \\
&\quad - \sum_{\substack{\rm self-intersections\\ (C_1)}} I[\ell]\Big[\langle\langle W_{\rm d}[C_1'] W_{\rm d}[C_2]\rangle\rangle\, \langle W_{\rm d}[C_1'']\rangle + (C_1' \leftrightarrow C_1'')\Big] + \\
&\quad + (C_1 \leftrightarrow C_2)\Bigg] - \\
&\quad - \sum_{\substack{\rm parallel\ gauge\\ \rm mutual\ intersections\\ (C_1,C_2)}} J[\ell]\langle W_{\rm d}[C_1 C_2]\rangle - \sum_{\substack{\rm anti-parallel\\ \rm mutual\ intersections\\ (C_1,C_2)}} K[\ell]\,\langle W_{\rm d}[C_1(\ell\bar{\ell})^{-1}C_2]\rangle + \\
&\quad + \sum_{\substack{\rm parallel\ gauge\\ \rm mutual\ intersections\\ (\widetilde{C}_1,C_2)}} J[\ell]\langle W_{\rm d}[\widetilde{C}_1 C_2]\rangle + \sum_{\substack{\rm anti-parallel\\ \rm mutual\ intersections\\ (\widetilde{C}_1,C_2)}} K[\ell]\langle W_{\rm d}[\widetilde{C}_1(\ell\bar{\ell})^{-1}C_2]\rangle + \\
&\quad + O\left(\frac{1}{N_{\rm c}^2}\right).
\end{aligned} \tag{4.33}$$

The coefficients $\widetilde{\beta}_{\ell,p}$, $s_{\ell,p}$, $I[\ell]$, $J[\ell]$ and $K[\ell]$ are all the same as defined previously [c.f., Eqs (3.19)–(3.27)]. Therefore, given the appropriate parent/daughter operator mapping and the above assumptions concerning symmetry realizations, the large $N_{\rm c}$ loop equations

argued to vanish in the parent theory. In the daughter theory, these terms correspond to splittings of the original loop C (on the extended lattice $\bar{\Lambda}$) into subloops C' and C'' each of which represent observables containing matter insertions whose $\mathbf{r}$ charges do not sum to zero. Such observables are not gauge-invariant in the daughter theory and their expectation values (in the gauge invariant measure of the theory) necessarily vanish.

for both single trace expectation values and two loop correlators coincide between parent and daughter orbifold theories.[32]

5. Discussion

The loop equations for single-trace expectation values (4.31) or correlators (4.33) may be solved iteratively (as described in section 3) to generate expansions in powers of the plaquette weights $\widetilde{\beta}^{(\mathrm{d})}_{\ell,p}$, or equivalently, double expansions in powers of the hopping parameter κ and the inverse 't Hooft coupling β_p/N_c. Consequently, the equality of large N_c loop equations between parent and daughter theories (under the appropriate operator mapping between the two theories) implies that expectation values or correlators of corresponding operators in the two theories have identical strong coupling/large mass expansions (in the large N_c limit). Standard methods for proving convergence of cluster expansions [25, 26, 27] may be generalized without difficulty to theories with product gauge groups such as the orbifold theories under consideration, and show that the strong coupling/large mass expansions in both parent and daughter theories have non-zero radii of convergence. As a result, equality of the strong coupling/large mass expansions immediately implies equality, within the radius of convergence, of the exact expectation values or correlators themselves. (This is why equality of strong coupling expansions is a much stronger result than equality of weak coupling perturbation theory.) And equality within the radius of convergence immediately extends, via analytic continuation, to exact equality throughout the portion of the phase diagram in both theories which is continuously connected to the strong coupling/large mass region.

As emphasized in the previous section, in order for the loop equations of parent and daughter orbifold theories to coincide, the parent theory must not spontaneously break the global symmetries used in the orbifold projection, and the daughter theory must not spontaneously break the discrete translation symmetry of theory space. Within the strong coupling/large mass phase of either theory, this is not an additional assumption; the convergence of the strong coupling/large mass expansion can easily be used to show that the $\mathrm{U}(N_\mathrm{s}) \times \mathrm{U}(N_\mathrm{f})$ global symmetry in the parent theory, and the $(Z_k)^d$ discrete symmetry in the daughter theory, are unbroken within this phase.

It should be noted that the large-N_c equivalence between parent and daughter orbifold theories which we have demonstrated (within the strong coupling/large mass phase of both theories) implies, in the large N_c limit, equality of the string tensions of the two theories as well as equality of their spectrum of excitations (within symmetry channels to which the parent/daughter operator mapping applies, namely channels invariant under the global parent symmetries used in the orbifold projection, and under theory space translations in the daughter). This merely reflects the fact that the string tension can be extracted from large Wilson loops, and the mass spectrum from the large distance behavior of correlators.

[32] Although we have focused on bosonic observables up to now, this assertion about the correspondence between parent and daughter loop equations for two loop correlators is true for fermionic as well as bosonic loops; see appendix A.

A further consequence of this large-N_c equivalence is the existence of relations between correlation functions of the daughter theory which reflect symmetries that are present in the parent theory, but absent in the daughter theory. For example, if differing **r**-charges are assigned to the set of scalar fields in a given orbifold projection, or differing **r**-charges are assigned to the set of fermions, then the daughter theory will not be invariant under the global $U(N_s) \times U(N_f)$ symmetry of the parent theory; instead the daughter theory will only be invariant under whatever subgroup of $U(N_s) \times U(N_f)$ preserves the **r**-charge assignments. However, the large N_c equivalence with the parent theory means that daughter theory correlation functions (in symmetry channels to which the parent/daughter operator mapping applies) will satisfy various $U(N_s) \times U(N_f)$ symmetry relations in the large N_c limit. This implies that the particle spectrum of the large N_c daughter theory must have degeneracies which do not follow from the symmetries of the daughter theory — but reflect projections of symmetry relations in the parent theory [13].[33]

Extending our results to a wider class of theories should be straightforward, but will be left to future work. Possible extensions include consideration of more general orbifold projections (such as cases where the projection-defining subgroup H is non-abelian), inclusion of Yukawa couplings, matter sectors with less flavor symmetry, other gauge groups [$O(N_c)$ or $Sp(N_c)$],[34] symmetric or antisymmetric tensor (instead of adjoint representation) matter fields [36], and alternative fermion discretizations.

As our method of proof (for the strong coupling/large mass phase of these lattice theories) makes clear, large N_c equivalence between parent and daughter orbifold theories has nothing whatsoever to do with supersymmetry, dimensionality, continuum limits, or large volume limits. However, the extent to which this non-perturbative equivalence holds outside the strong coupling/large mass phase is not yet clear. Large N_c equivalence between parent and daughter theories clearly fails to hold in any phase of the parent theory which spontaneously breaks the particular global symmetries used in the orbifold projection, as well as in any phase of the daughter theory which spontaneously breaks the discrete theory space translation symmetry. Such phases (when they exist) do not have equivalent loop equations.[35] But as long as these symmetries are not spontaneously broken, then the loop equations of the two theories coincide. The only way large N_c equivalence could fail in this circumstance is if there are multiple physically acceptable solutions to the loop

[33]A simple example is a parent theory containing two fermions, and a U(2) flavor symmetry. A Z_2 orbifold projection with zero r-charge for one fermion and unit r-charge for the other yields a daughter theory with one set of adjoint fermions (χ^j), one set of bifundamentals (ψ^j), and only a $U(1) \times U(1)$ flavor symmetry. The symmetry relations between correlators of the parent theory which survive projection to the daughter theory require, for example, that the two-point functions of $\sum_j \mathrm{tr}(\chi^j \chi^j)$ and $\sum_j \mathrm{tr}(\psi^j \psi^{j+1})$ coincide, implying degeneracy between the masses of single particle states containing two adjoint fermions and those with two bifundamentals.

[34]It should be noted that $SU(N_c)$ and $U(N_c)$ gauge theories have coinciding large N_c limits; excluding (or including) the central U(1) factor only affects subleading $1/N_c^2$ suppressed contributions to either single-trace expectation values or connected multi-loop correlators.

[35]The Z_2 orbifold of super-Yang-Mills theory compactified on $R^3 \times S^1$ is an example of an orbifold theory with a phase in which the theory space translation symmetry is spontaneously broken (as shown by Tong [17]), thereby invalidating the large N_c equivalence in this phase.

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equations, and the parent and daughter theories correspond to different solutions. As mentioned in section 2, for simple models involving only a few plaquettes, it is known that supplementing the loop equations by trivial inequalities (reflecting unitarity of the gauge connection) allows one to select the correct solution of the loop equations on the weak coupling side of the Gross-Witten large N_c phase transition. This may well be true more generally. In any case, extending our loop-equation based proof of large N_c equivalence to other phases of lattice gauge theories, including weak coupling phases with physical continuum limits, will require better understanding of when (or if) parent and daughter theories can correspond to different solutions of the same set of loop equations.

It is quite possible that a stronger version of our results, valid beyond the strong coupling/large mass phase, may be obtained by comparing the large N_c coherent state variational actions [23, 37] of the parent and daughter theories. The minimum of this variational action yields the free energy in the large N_c limit. The loop equations for single trace operators are, in effect, equations characterizing the location of stationary points of this large N_c variational action, but the value of the variational action itself is needed to determine which stationary point describes the correct equilibrium state of the theory. This is a topic for future work.

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A. Correlators of fermionic observables

Decorated Wilson loops containing an odd number of fermion insertions do not correspond to closed loops on the extended lattice $\bar{\Lambda}_f$, as it was defined in section 3.2. Instead they correspond to open contours in $\bar{\Lambda}_f$ whose endpoints are Z_2 partners of each other. This is a perfectly consistent representation of single-trace fermionic observables, although it has the drawback of involving a distinguished starting (and ending) site on the loop. In other words, this representation does not make trace cyclicity manifest.

The previous construction of the extended lattice $\bar{\Lambda}_f$ was dictated by the desire to represent, geometrically, bosonic observables containing an even number of fermion insertions. Such observables only satisfy trace cyclicity up to a sign, which is why it was necessary for the extended lattice $\bar{\Lambda}_f$ to involve a doubling of sites in the original lattice Λ (so that a loop and its Z_2 'mirror' will represent the same observable, but with opposite overall signs). In contrast, fermionic single-trace observables *do* satisfy trace cyclicity, without any minus signs (because moving a fermion insertion from one end to the other now involves an even number of fermionic transpositions). So for fermionic observables, an alternate, and simpler, geometric encoding is to represent these observables as ordinary closed loops on the smaller extended lattice $\bar{\Lambda}_{\rm f}/Z_2$, in which all Z_2 partner sites s and $\tilde{s}$ in $\bar{\Lambda}_{\rm f}$ are now identified. (Hence the result looks just like the minimally extended lattice $\bar{\Lambda}_{\rm s}$ for scalars).

Using either representation, one may generalize the previous treatment of loop equations for decorated loops, described in section 3.3, to the case of fermionic loops. All expectation values of fermionic loops vanish,[36]so the only relevant fermionic loop equations are those for connected correlators involving an even number of fermionic loops. For two-loop correlators, the result may be written in the form (3.25) previously derived for bosonic loops, except that there is no need to include the mutual intersection terms involving Z_2 shifted loops if the $\bar{\Lambda}_{\rm f}/Z_2$ representation is used.[37]

Considering correlators of fermionic operators instead of bosonic operators makes no difference as far the equivalence between parent and daughter orbifold loop equations is concerned. This presumes, of course, that gauge invariant fermionic operators exist in the daughter theory; this will depend on the chosen **r**-charge assignments. If the chosen **r**-charges do allow fermionic single trace operators in the daughter theory, then the loop equations for connected correlators of fermionic loops coincide (under the parent/daughter operator mapping) under the same conditions needed for coinciding bosonic correlators — the parent theory must not spontaneously break the symmetries used in the orbifold projection and the daughter theory must not spontaneously break its theory space translation symmetry.

B. Iterative solution of loop equations

It is instructive to see how the physics of confinement emerges directly from the loop equations in the strong coupling limit of a lattice gauge theory.[38] For simplicity, consider a pure $\mathrm{U}(N_{\rm c})$ gauge theory formulated on a simple cubic lattice, and take the coupling $\widetilde{\beta} \equiv \beta_p/N_{\rm c}$ to be the same for all plaquettes. To generate the strong coupling (small $\widetilde{\beta}$) expansion of Wilson loop expectation values, one may imagine assembling all possible loops into an (infinitely) long vector, and then repeatedly iterating the the loop equations (2.17) for this vector, starting with zero expectation values for all loops except the trivial loop $\langle 1 \rangle = 1$. After a single iteration, one finds that the only loops with $O(\widetilde{\beta})$ expectations are elementary plaquettes,

$$\langle \square \rangle = \frac{\widetilde{\beta}}{2} \langle 1 \rangle + (\text{higher-order})\,. \tag{B.1}$$

[36]One might wonder if it is ever possible to break spontaneously the Z_2 symmetry [often called $(-1)^F$] which distinguishes fermions from bosons. We will ignore this perverse possibility.

[37]More precisely, one should ignore the -1 factors in $I[\ell]$ and $J[\ell]$ associated with intersection links starting on $\widetilde{\Lambda}$, and redefine $K[\ell]$ to be -1 for gauge, scalar, or backward-directed fermion (ψ) links, and $+1$ for forward-directed fermion ($\bar{\psi}$) links. In the self intersection and mutual intersection terms, loops appearing in single trace expectation values (which correspond to bosonic observables) should be "lifted" to the original extended lattice $\bar{\Lambda}_{\rm f}$ by regarding the intersection link ℓ as starting from a site in the physical lattice Λ.

[38]There are arguably more direct methods for generating the strong coupling expansion of lattice gauge theories [38, 26]. The point of this appendix is merely to illustrate how the minimal set of loop equations (2.17) suffice for extracting this physics.

On the right-hand side of (B.1), the trivial loop appears in the deformation terms, which remove a plaquette from the original loop. Other deformation terms, which attach a plaquette to the original loop, vanish at this order in the iteration (as well as the next), and lead to higher order corrections [of order $\widetilde{\beta}^3$]. At second order in the iteration, one finds $O(\widetilde{\beta}^2)$ expectation values for two-plaquette loops. For example,

$$\langle \boxed{} \rangle = \left(\frac{\widetilde{\beta}}{2}\right)^2 + (\text{higher-order})\,. \tag{B.2}$$

One may see directly from the loop equations that every plaquette deformation of a loop is associated with one factor of $\widetilde{\beta}$. Consequently, for a general loop of area A (i.e., a loop whose minimal spanning surface contains A plaquettes), it is easy to see that one must iterate the loop equations A times before generating a non-zero contribution, so that $\langle W[C]\rangle = O(\widetilde{\beta}^A)$. Determining the coefficient is easy once one realizes that the number of deformation terms leading to decrease in the area of a loop [in the minimal loop equations (2.17)] precisely equals the number of links forming the loop. These terms give identical contributions (at leading order) and, in effect, cancel the factor of the loop perimeter on the left side of the loop equations. Consequently, one finds confining area-law behavior,

$$\langle W[C]\rangle = \left(\frac{\widetilde{\beta}}{2}\right)^A + (\text{higher-order})\,, \tag{B.3}$$

or $\langle W[C]\rangle \sim e^{-\sigma A}$ with a string tension (in lattice units)

$$\sigma = \ln \frac{2}{\widetilde{\beta}}\,, \tag{B.4}$$

up to sub-leading corrections. With a bit more effort, one may show that corrections to the string tension are $O(\widetilde{\beta}^4)$.

It is straightforward to repeat the analysis when adjoint matter fields are present. If both $\widetilde{\beta}$ and κ/m are small (corresponding to strong coupling and large mass), then a similar iteration of the loop equations shows that Wilson loops still exhibit area law behavior (which would not be the case, of course, with fundamental representation matter fields). Dynamical adjoint matter fields only generate contributions to the string tension which are suppressed by at least κ^6,

$$\sigma = \ln\left(\frac{2}{\widetilde{\beta}}\right) + O(\widetilde{\beta}^4) + O\left(\frac{\kappa^6}{m^6}\right)\,. \tag{B.5}$$

A similar iterative approach may be applied to the loop equations (2.23) for connected correlators of Wilson loops. Consider, for example, the correlator of two elementary plaquettes separated by a lattice distance L, $\langle\langle \square \xleftrightarrow{\;L\;} \square \rangle\rangle$. Iterating the loop equations (2.23), starting with all two-loop connected correlators equal to zero, one may easily see that non-zero contributions will only arise after some sequence of plaquette deformations acting on one or the other plaquette (or both) causes the deformed loops to have a mutual intersection. Since each plaquette deformation costs a factor of $\widetilde{\beta}$, the leading contribution must involve a power of $\widetilde{\beta}$ which is proportional to L. Determining the correct power (directly from the loop equations) is a bit tricky. After $L+1$ iterations of the loop equations, one first finds mutual intersection terms of the form

$$\langle\langle \square \overset{L}{\longleftrightarrow} \square \rangle\rangle \sim \widetilde{\beta}^{L} \langle \overset{L}{\boxed{\quad\quad}} \rangle + (\text{other terms})\,. \tag{B.6}$$

The Wilson loop on the right has an $O(\widetilde{\beta}^{L+2})$ expectation value, so one might expect the two plaquette connected correlator to be $O(\widetilde{\beta}^{2L+2})$. However, there are cancellations between deformation and mutual intersection terms, which eliminate all contributions below order $\widetilde{\beta}^{4L}$. A more careful analysis shows that the first non-zero contribution comes from deformations which build a "tube" between the two initial plaquettes, so that[39]

$$\langle\langle \square \overset{L}{\longleftrightarrow} \square \rangle\rangle = \left(\frac{\widetilde{\beta}}{2}\right)^{4L} \times [1 + O(\widetilde{\beta}^2)]\,. \tag{B.7}$$

Consequently, the correlator falls exponentially with distance, $\langle\langle \square \overset{L}{\longleftrightarrow} \square \rangle\rangle \sim e^{-\mu L}$, with a mass gap μ (equal to the lightest glueball mass) given by

$$\mu = 4 \ln \frac{2}{\widetilde{\beta}}\,, \tag{B.8}$$

up to sub-leading corrections [which turn out to be $O(\widetilde{\beta}^4)$]. As with the string tension, inclusion of adjoint matter fields only produces at most $O(\kappa^6)$ sub-leading corrections to the mass gap μ.

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[39] This result is valid in three or more dimensions. In two dimensions, the plaquette-plaquette correlation function vanishes identically (except when the plaquettes coincide).

One can see this from the fact that in d=2, a suitable choice of gauge allows one to rewrite the integral over link variables as an integral over independent plaquette variables. The resulting partition function factorizes into a product of single plaquette contributions, and correlations between different plaquettes are absent.

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